

CASTELNUOVO-MUMFORD REGULARITY AND GV-SHEAVES ON IRREGULAR VARIETIES

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ABSTRACT. Inspired by Beauville's recent construction of Ulrich sheaves on abelian surfaces [Bea], we pose the question of whether a torsion-free sheaf on a polarized smooth projective variety with Castelnuovo-Mumford regularity 1 is a GV (generic vanishing) sheaf, and present evidence that this question is governed by the positivity of curves on generalized Brill-Noether loci. We prove that it has an affirmative answer for natural polarizations on many well-known irregular surfaces, as well as some polarizations on ruled threefolds over a curve.

INTRODUCTION

Castelnuovo-Mumford regularity measures the homological complexity of a graded module, and is therefore central to understanding coherent sheaves on \mathbb{P}^N . If we restrict our attention to sheaves supported on an *irregular* smooth subvariety $X \subset \mathbb{P}^N$, the Fourier-Mukai methods developed by Pareschi and Popa in [PP1, PP3, PP4] can also be brought to bear. In this note, we investigate how the Castelnuovo-Mumford regularity of such sheaves interacts with this "Fourier-Mukai geometry."

The interactions described by our results depend on whether the generalized Brill-Noether locus associated to $\mathcal{O}_X(1)$ contains a sufficiently positive curve. It is classically known that such curves exist when $\dim(X) = 1$; the picture when $\dim(X) \geq 2$ is not yet clear, even though some dimension estimates exist for generalized Brill-Noether loci [CP, MPP]. We expect recent advances in the positivity theory of cycles (e.g. [FL]) to play a role in further progress on the questions raised here.

Turning to details, if X is a smooth projective variety, $\mathcal{O}_X(1)$ is an ample and globally generated line bundle on X , and \mathcal{F} is a coherent sheaf on X supported in dimension ≥ 1 , the *CM (Castelnuovo-Mumford) regularity* of \mathcal{F} on X with respect to $\mathcal{O}_X(1)$ is defined as

$$(0.1) \quad \text{reg}_{\mathcal{O}_X(1)}(\mathcal{F}) := \min\{m \in \mathbb{Z} : \forall i > 0 \ H^i(\mathcal{F}(m-i)) = 0\}$$

In addition to measuring the complexity of \mathcal{F} , CM-regularity also helps measure the positivity of \mathcal{F} ; the Castelnuovo-Mumford lemma implies that $\mathcal{F}(m)$ is globally generated for all $m \geq \text{reg}_{\mathcal{O}_X(1)}(\mathcal{F})$, while $\mathcal{F}(m)$ may not have any global sections at all if $m < \text{reg}_{\mathcal{O}_X(1)}(\mathcal{F})$. In what follows, a polarization is understood to be a line bundle which is globally generated as well as ample.

We are interested in the CM-regularity of a coherent sheaf \mathcal{F} on X in the case $H^1(\mathcal{O}_X) \neq 0$, where the *cohomological support loci*

$$(0.2) \quad V^i(\mathcal{F}) := \{\alpha \in \text{Pic}^0(X) : H^i(\mathcal{F} \otimes \alpha) \neq 0\}, \ 0 \leq i \leq \dim(X)$$

are fundamentally important. The dimensions of the $V^i(\mathcal{F})$ are invariant under tensoring by elements of $\text{Pic}^0(X)$; however, the CM regularity of \mathcal{F} is generally not. We define the *continuous CM-regularity of \mathcal{F} with respect to $\mathcal{O}_X(1)$* as

$$(0.3) \quad \text{reg}_{\mathcal{O}_X(1)}^{\text{cont}}(\mathcal{F}) := \min\{m \in \mathbb{Z} : \forall i > 0 \ V^i(\mathcal{F}(m-i)) \neq \text{Pic}^0(X)\}$$

This gives a slightly coarser measure of positivity than CM-regularity; see Lemma 1.5 for a precise statement. The structure of cohomological support loci was addressed in [GL] and [Ha], and their connections with positivity were pursued in [De, PP1, PP4]. A key notion in these papers, as well as this note, is the following: \mathcal{F} is said to be a *GV-sheaf* if $\text{codim}(V^i(\mathcal{F})) \geq i$ for all $i > 0$. This property can be viewed as a weak form of positivity in at least one sense; when the Albanese map of X is finite, the stronger condition that $\text{codim}(V^i(\mathcal{F})) > i$ for all $i > 0$ (this is known as *M-regularity*) implies that \mathcal{F} is ample ([De], Corollary 3.2).

In recent years a special class of sheaves with CM-regularity 0 on a polarized variety of dimension $n \geq 1$ has been intensely studied; this is the class of *Ulrich bundles*, i.e. vector bundles whose twist by $-i$ has no cohomology for $1 \leq i \leq n$. Very recently, Beauville produced a family of rank-2 Ulrich bundles on abelian surfaces via the Serre method [Bea]. Each rank-2 bundle \mathcal{F} arising from his construction has the property that $\mathcal{F}(1)$ is Ulrich; in particular \mathcal{F} has CM-regularity 1. It can be verified directly that this \mathcal{F} is a GV-sheaf. On a different note, every polarized curve admits an Ulrich bundle \mathcal{E} (e.g. [ESW]) and the vanishing $H^1(\mathcal{E}(-1)) = 0$ implies that $\mathcal{E}(-1)$ is a GV-sheaf in this case.

Since the Ulrich property is an open condition on families of vector bundles, it follows that in general the twist of any Ulrich bundle by -1 has continuous CM-regularity equal to 1. Given that irrational curves and abelian surfaces are the only irregular varieties currently known to admit Ulrich bundles, one can ask if the twist of an Ulrich bundle by -1 is always a GV-sheaf. Our results address the following question, which is broader in scope.

(\star) Let X be a smooth projective variety of dimension $n \geq 1$ and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X . If \mathcal{F} is a torsion-free sheaf on X satisfying $\text{reg}_{\mathcal{O}_X(1)}^{\text{cont}}(\mathcal{F}) \leq 1$, is \mathcal{F} a GV-sheaf?

Note that we trivially have an affirmative answer when $H^1(\mathcal{O}_X) = 0$. One of the motivations for the study of M-regularity and related concepts in [PP1] was to understand subvarieties of abelian varieties via the kind of insight that Castelnuovo-Mumford regularity provides for subvarieties of projective space. Theorems of this type can be found in [PP1, PP2, PP3] and more recently in [LN]. Our inquiries point in a different direction, since we are interested in deducing generic vanishing statements from "honest" Castelnuovo-Mumford regularity.

Our first result highlights the importance of the generalized Brill-Noether locus $V^0(\mathcal{O}_X(1))$. Even though the conclusion holds under a weaker hypothesis on \mathcal{F} , the present phrasing emphasizes the connection with (\star).

Theorem A. *Let X be a smooth projective variety of dimension $n \geq 1$, and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X satisfying the property that $V^0(\mathcal{O}_X(1))$ contains a curve T whose numerical class is a proportional to a power of an ample divisor class on $\text{Pic}^0(X)$. If \mathcal{F} is a torsion-free sheaf on X satisfying $\text{reg}_{\mathcal{O}_X(1)}^{\text{cont}}(\mathcal{F}) \leq 1$, then $\text{codim}(V^n(\mathcal{F})) \geq n$.*

The hypothesis on $\mathcal{O}_X(1)$ is trivially satisfied whenever $V^0(\mathcal{O}_X(1)) = \text{Pic}^0(X)$. An Euler characteristic calculation shows that this holds if $h^0(\mathcal{O}_X(1)) > 1$ and $\mathcal{O}_X(1)$ is a GV-sheaf; by Corollary C of [PP4], any globally generated adjunction of a nef line bundle has this property when X has maximal Albanese dimension. See Remark 2.2 for a family of examples satisfying $V^0(\mathcal{O}_X(1)) \neq \text{Pic}^0(X)$.

The next result follows easily from the proof of Theorem A (which will be discussed momentarily) but its assumption on $\mathcal{O}_X(1)$ is much weaker.

Theorem B. *Let X be a smooth projective surface with $h^1(\mathcal{O}_X) > 0$, and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X such that $V^0(\mathcal{O}_X(1))$ contains a curve T whose numerical class lies in the interior of the cone of curves of $\text{Pic}^0(X)$. Then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$.*

The following consequence, immediate from Kawamata-Viehweg vanishing, implies that if \mathcal{E} is any Ulrich bundle on a polarized abelian surface then $\mathcal{E}(-1)$ is a GV-sheaf. It is worth noting that Beauville's Ulrich bundles, for which this can be checked explicitly, only account for a hypersurface in the relevant moduli space (Remark 3, [Bea]).

Corollary C. *If X is a smooth projective surface and \mathcal{L} is a nef and big line bundle on X for which $\mathcal{O}_X(1) := \omega_X \otimes \mathcal{L}$ is ample and globally generated, then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$.*

We also use Theorem B to obtain a positive answer to (\star) for some natural polarizations on Cartesian and symmetric products of curves (Propositions 3.1 and 3.2, respectively) and for all polarizations on some surfaces isogenous to a product of curves (Proposition 3.3).

The idea behind our proofs of Theorems A and B is to use the curve $T \subseteq V^0(\mathcal{O}_X(1))$ to construct a "large enough" positive cycle in $\text{Pic}^0(X)$ that does not intersect $V^n(\mathcal{F})$. For Theorem A, this cycle is a Pontryagin product of T (Definition 1.9), and we use a calculation from [DELV] to show that its numerical class is proportional to a product of ample divisors (Lemma 1.10). Our construction suggests a roadmap for settling (\star) in the affirmative: take a sufficiently positive curve T in $\text{Pic}^0(X)$ and for $1 \leq i \leq n-1$, prove that the $(i-1)$ st Pontryagin product of T intersects every effective cycle of codimension $i-1$ but does not intersect $V^i(\mathcal{F})$. Up to now we have been unable to carry this out in full generality. However, we have verified (\star) for some scrollar embeddings of ruled threefolds over a curve (Proposition 4.1).

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1. PRELIMINARIES

Throughout, we work over an algebraically closed field of characteristic zero. In what follows, X is a smooth projective variety of dimension $n \geq 1$, $\mathcal{O}_X(1)$ is an ample and globally generated line bundle on X , and \mathcal{F} is a coherent sheaf on X supported in dimension ≥ 1 .

1.1. Continuous CM-regularity. We discuss the definition (0.3) in more detail.

Definition 1.1. If $k \in \mathbb{Z}$, we say \mathcal{F} is continuously k -regular with respect to $\mathcal{O}_X(1)$ if for $1 \leq i \leq n$ we have that $V^i(\mathcal{F}(k-i)) \neq \text{Pic}^0(X)$.

Lemma 1.2. *The following are equivalent:*

- (i) $\mathcal{F} \otimes \alpha$ is continuously k -regular with respect to $\mathcal{O}_X(1)$ for all $\alpha \in \text{Pic}^0(X)$.
- (ii) \mathcal{F} is continuously k -regular with respect to $\mathcal{O}_X(1)$.
- (iii) $\mathcal{F} \otimes \alpha$ is continuously k -regular with respect to $\mathcal{O}_X(1)$ for some $\alpha \in \text{Pic}^0(X)$.
- (iv) $\mathcal{F} \otimes \alpha$ is k -regular with respect to $\mathcal{O}_X(1)$ in the sense of Castelnuovo-Mumford for some $\alpha \in \text{Pic}^0(X)$. □

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are all immediate. For (iv) \Rightarrow (i), observe that for all i the dimension of $V^i(\mathcal{F}(k-i))$ is invariant under tensoring by elements of $\text{Pic}^0(X)$. \square

The next statement follows from combining (iv) of Lemma 1.2 with the corresponding property of Castelnuovo-Mumford regularity.

Corollary 1.3. *If \mathcal{F} is continuously k -regular, then \mathcal{F} is continuously k' -regular for all $k' \geq k$.* \square

Definition 1.4. The continuous CM-regularity of \mathcal{F} with respect to $\mathcal{O}_X(1)$ is the smallest integer k for which \mathcal{F} is continuously k -regular with respect to $\mathcal{O}_X(1)$; we denote it by $\text{reg}_{\mathcal{O}_X(1)}^{\text{cont}}(\mathcal{F})$.

We conclude this subsection with evidence for our earlier assertion that continuous CM-regularity helps measure positivity. Recall that a coherent sheaf \mathcal{F} on X is *continuously globally generated* if there is a nonempty Zariski-open subset $U \subseteq \text{Pic}^0(X)$ such that the evaluation map

$$(1.1) \quad \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \rightarrow \mathcal{F}$$

is surjective. The following fact is immediate from (iv) of Lemma 1.2.

Lemma 1.5. *If \mathcal{F} is continuously k -regular with respect to $\mathcal{O}_X(1)$, then $\mathcal{F}(k)$ is continuously globally generated. In particular, $\mathcal{F}(k)$ is nef.* \square

Remark 1.6. One of the main results in [PP1] is that the M-regularity of a coherent sheaf on an abelian variety implies continuous global generation. In light of Lemma 1.5 and the line of inquiry suggested by (\star) , it is natural to ask whether 0-regularity with respect to $\mathcal{O}_X(1)$ implies M-regularity.

1.2. Numerical Cycle Classes and Pontryagin Products. We now collect some statements on intersections of cycles that will be used in the sequel. In what follows, X is a smooth projective variety of dimension $n \geq 1$.

Definition 1.7. For $0 \leq k \leq n$, $N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$, where $N_k(X)_{\mathbb{Z}}$ is the group of algebraic k -cycles modulo numerical equivalence. The numerical dual group $N^k(X)$ is the dual of $N_k(X)$.

Since the evaluation pairing $N^k(X) \times N_k(X) \rightarrow \mathbb{R}$ can be identified with an intersection pairing, we view $N^k(X)$ as parametrizing numerical classes of cycles of codimension k . The following result on intersections of divisors will be used in the proof of Proposition 3.1.

Lemma 1.8. *Let Y and Z be projective varieties of respective dimensions m and n , and let π_Y and π_Z be the projections from $Y \times Z$ to Y and Z . Let D be a nef divisor on $Y \times Z$, and let H, H_Y, H_Z be ample divisors on $Y \times Z, Y$ and Z , respectively. Then if $D \cdot H \cdot \pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^{n-1} = 0$ we have $D = 0$.*

Proof. The ampleness of H_Y and H_Z implies that $\pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^{n-1}$ is proportional to the numerical class of a surface in $Y \times Z$. Since D is nef, it is a limit of ample divisor classes, so $D \cdot \pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^{n-1}$ is in the closed cone of curves in $N_1(Y \times Z)$. Given that its intersection with H is zero by hypothesis, the fact that H is in the interior of the nef cone of X implies that $D \cdot \pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^{n-1} = 0$.

It follows at once that $D \cdot \pi_Y^* H_Y^m \cdot \pi_Z^* H_Z^{n-1} = D \cdot \pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^n = 0$. Consequently, we have

$$D \cdot (\pi_Y^* H_Y + \pi_Z^* H_Z)^{m+n-1} = D \cdot \left(\binom{m+n-1}{m-1} \pi_Y^* H_Y^{m-1} \cdot \pi_Z^* H_Z^n + \binom{m+n-1}{m} \pi_Y^* H_Y^m \cdot \pi_Z^* H_Z^{n-1} \right) = 0$$

Since $\pi_Y^* H_Y + \pi_Z^* H_Z$ is ample, the class $(\pi_Y^* H_Y + \pi_Z^* H_Z)^{m+n-1}$ is proportional to the class of a complete intersection curve on $Y \times Z$, and therefore lies in the interior of the cone of curves in $N_1(Y \times Z)$, so that our nef divisor D is 0. \square

Specializing to the case where X is an abelian variety, we now introduce the main tool in the proof of Theorem A.

Definition 1.9. If X is an abelian variety of dimension $g \geq 1$, T is a subvariety of X , and $1 \leq k \leq g$, the k -th Pontryagin product $T^{*(k)}$ of T is the image of T^k under the addition map $\sigma^k : X^k \rightarrow X$.

Note that when $\dim(T) = 1$, the dimension of $T^{*(k)}$ is equal to $\min\{k, g\}$. The following result is an immediate consequence of Lemma 1.9(a) in [DELV].

Lemma 1.10. *Let X be an abelian variety of dimension $g \geq 2$, and let $\alpha \in N_k(X)$ be given, where $1 \leq k \leq g-1$. If H is an ample divisor class on X and $T \subset X$ is a curve whose numerical class is proportional to H^{g-1} , then for $1 \leq k \leq g-1$ the numerical class of $T^{*(k)}$ is a positive multiple of H^{g-k} .* \square

Proposition 1.11. *For $1 \leq d \leq g$ and $\mathcal{L} \in \text{Pic}^d(C)$, the numerical class of $V^0(\mathcal{L})$ in $N^{g-d}(\text{Pic}^0(C))$ is a positive rational multiple of θ^{g-d} , where $\theta \in N^1(\text{Pic}^0(C))$ is the numerical class of the theta-divisor, and the image $V^0(\mathcal{L})^-$ of $V^0(\mathcal{L})$ under inversion on $\text{Pic}^0(C)$ is numerically equivalent to $V^0(\mathcal{L})$.*

Proof. The statement about the numerical class of $V^0(\mathcal{L})$ follows from Theorem (1.3) on p. 212 of [ACGH], and the invariance of this class under inversion follows from the invariance of θ under inversion. \square

2. PROOFS OF THEOREMS A AND B

Theorem A is an immediate consequence of the following stronger statement.

Proposition 2.1. *Let X be a smooth projective variety of dimension $n \geq 2$ and irregularity $q \geq 1$, and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X such that $V^0(\mathcal{O}_X(1))$ contains a curve T algebraically equivalent to a complete intersection of algebraically equivalent ample divisors on $\text{Pic}^0(X)$. Then if \mathcal{F} is a torsion-free sheaf on X for which $H^n(\mathcal{F}(1-n)) = 0$, we have that $\text{codim}(V^n(\mathcal{F})) \geq n$.*

Proof. We consider the $(n-1)$ st Pontryagin product of T , which has the set-theoretic description

$$(2.1) \quad T^{*(n-1)} = \{\alpha_1 \otimes \cdots \otimes \alpha_{n-1} : \alpha_1, \dots, \alpha_{n-1} \in T\}$$

Fix $\bar{\alpha} := \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \in T^{*(n-1)}$. Then for each j there exists an effective divisor $D_j \in |\mathcal{O}_X(1) \otimes \alpha_j|$. If $D_{\bar{\alpha}} := \sum_j D_j$, we have an exact sequence

$$(2.2) \quad 0 \rightarrow \mathcal{F}(1-n) \rightarrow \mathcal{F} \otimes \bar{\alpha} \rightarrow \mathcal{F}|_{D_{\bar{\alpha}}} \otimes \bar{\alpha} \rightarrow 0$$

It follows at once from our hypothesis on \mathcal{F} that $H^n(\mathcal{F} \otimes \bar{\alpha}) = 0$. Letting $\bar{\alpha}$ vary over $T^{*(n-1)}$, we conclude that $V^n(\mathcal{F}) \cap T^{*(n-1)} = \emptyset$.

There are now two cases to consider. If $n \leq q$, then by Lemma 1.10 and our hypothesis on T , we have that $T^{*(n-1)}$ is an $(n-1)$ -dimensional subvariety of $\text{Pic}^0(X)$ whose cycle class is proportional to a product of ample classes by Lemma 1.10. In particular, $T^{*(n-1)}$ must intersect every subvariety of $\text{Pic}^0(X)$ having codimension at most $n-1$. It follows that $\text{codim} V^n(\mathcal{F}) \geq n$. On the other hand, if $n > q$, then $T^{*(n-1)} = \text{Pic}^0(X)$, so that $V^n(\mathcal{F}) = \emptyset$. \square

Remark 2.2. For every $n \geq 2$ there is an n -dimensional variety X and a line bundle $\mathcal{O}_X(1)$ on X satisfying the hypothesis of Theorem A for which $V^0(\mathcal{O}_X(1)) \neq \text{Pic}^0(X)$. Let C be a smooth projective curve of genus $g \geq 6$, and let $X := C^{(n)}$ be its n -th symmetric product. The image of the injective map $f_n : \text{Pic}^0(C) \rightarrow \text{Pic}^0(C^n) \cong \text{Pic}^0(C)^n$ defined by $f_n(\alpha) = \alpha^{\boxtimes n}$ is the locus invariant under the action of the symmetric group, so f_n factors canonically through an isomorphism $f_{(n)} : \text{Pic}^0(C) \rightarrow \text{Pic}^0(X)$. If $\mathcal{O}_C(1)$ is an ample and globally generated line bundle on C of degree $d < g-3$, there is a line bundle $\mathcal{O}_X(1)$ on X whose pullback via the quotient map $\pi : C^n \rightarrow X$ is isomorphic to $\mathcal{O}_C(1)^{\boxtimes n}$. By the calculations in Section 6.1 of [Iz] we have for all $\alpha \in \text{Pic}^0(C)$ an isomorphism

$$(2.3) \quad H^0(\mathcal{O}_X(1) \otimes f_{(n)}(\alpha)) \cong \text{Sym}^n H^0(\mathcal{O}_C(1) \otimes \alpha).$$

It follows that restricting $f_{(n)}$ to $V^0(\mathcal{O}_C(1))$ induces an isomorphism $V^0(\mathcal{O}_C(1)) \cong V^0(\mathcal{O}_X(1))$. Since the numerical class of $V^0(\mathcal{O}_C(1))$ is proportional to θ^{g-d} by Proposition 1.11, intersecting $V^0(\mathcal{O}_C(1))$ with $d-1$ general divisors of numerical class 2θ produces a curve in $V^0(\mathcal{O}_X(1))$ whose numerical class is proportional to a power of an ample divisor.

Proof of Theorem B: By Lemma 1.2, we may assume without loss of generality that \mathcal{F} is 1-regular with respect to $\mathcal{O}_X(1)$. Then $H^1(\mathcal{F}) = 0$, and it is immediate that $\text{codim}(V^1(\mathcal{F})) \geq 1$.

Setting $n = 2$ in the proof of Proposition 2.1, we can also conclude that $V^2(\mathcal{F}) \cap T = \emptyset$. Since $\text{Pic}^0(X)$ is an abelian variety, its cones of nef and pseudoeffective divisors are equal, so the interiors of their duals in $N^{q-1}(\text{Pic}^0(X))$ coincide as well; consequently T must intersect every codimension-1 subvariety of $\text{Pic}^0(X)$. It follows that $\text{codim}(V^2(\mathcal{F})) \geq 2$. \square

3. PRODUCTS OF CURVES

We now give some applications of Theorem B. Observe that the hypothesis of the following result is satisfied when $\mathcal{O}_X(1)$ is the Segre product of ample and globally generated line bundles on C_1 and C_2 .

Proposition 3.1. *Let C_1 and C_2 be smooth projective curves of respective genera g_1 and g_2 , and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on $X := C_1 \times C_2$ such that $|\mathcal{O}_X(1)(-F_1 - F_2)|$ is nonempty whenever F_1 and F_2 are fibers of the projections from X to C_1 and C_2 , respectively. Then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$.*

Proof. It suffices to construct a curve in $V^0(\mathcal{O}_X(1))$ satisfying the hypothesis of Theorem B. For $i = 1, 2$ let $p_i : X \rightarrow C_i$ and $q_i : \text{Pic}^0(X) \cong \text{Pic}^0(C_1) \times \text{Pic}^0(C_2) \rightarrow \text{Pic}^0(C_i)$ be projection maps, and fix a point $x_i \in C_i$. If $F_i = p_i^{-1}(x_i)$, our hypothesis on $\mathcal{O}_X(1)$ implies that for $\alpha_1 \in V^0(\mathcal{O}_{C_1}(x_1))$ and $\alpha_2 \in V^0(\mathcal{O}_{C_2}(x_2))$ we have

$$(3.1) \quad 0 \neq H^0(\mathcal{O}_X(1)(-F_1 - F_2) \otimes p_1^* \alpha_1^\vee \otimes p_2^* \alpha_2^\vee) \subseteq H^0(\mathcal{O}_X(1) \otimes p_1^* \alpha_1^\vee \otimes p_2^* \alpha_2^\vee)$$

For each i the locus $V^0(\mathcal{O}_{C_i}(x_i))$ is the image of C_i under an Abel-Jacobi embedding in $\text{Pic}^0(C_i)$. The canonical identification of $\text{Pic}^0(C_1) \times \text{Pic}^0(C_2)$ with $\text{Pic}^0(X)$ is induced by the

map $(\beta_1, \beta_2) \mapsto p_1^* \beta_1 \otimes p_2^* \beta_2$, so if $V^0(\mathcal{O}_{C_i}(x_i))^-$ is the image of $V^0(\mathcal{O}_{C_i}(x_i))$ under inversion on $\text{Pic}^0(C_i)$ we have from (3.1) that

$$(3.2) \quad V^0(\mathcal{O}_{C_1}(x_1))^- \times V^0(\mathcal{O}_{C_2}(x_2))^- \cong q_1^{-1} V^0(\mathcal{O}_{C_1}(x_1))^- \cap q_2^{-1} V^0(\mathcal{O}_{C_2}(x_2))^- \subseteq V^0(\mathcal{O}_X(1)).$$

Let H be a very ample divisor on $\text{Pic}^0(X)$. Then the numerical cycle class defined by

$$(3.3) \quad \gamma := H \cdot q_1^*[V^0(\mathcal{O}_{C_1}(x_1))^-] \cdot q_2^*[V^0(\mathcal{O}_{C_2}(x_2))^-] \in N_1(X)$$

represents a curve $\Gamma \subseteq V^0(\mathcal{O}_X(1))$ obtained by intersecting the surface $q_1^{-1} V^0(\mathcal{O}_{C_1}(x_1))^- \cap q_2^{-1} V^0(\mathcal{O}_{C_2}(x_2))^-$ with a general divisor linearly equivalent to H . We will be done once we show that γ has positive intersection with every nonzero pseudoeffective divisor on $\text{Pic}^0(X)$.

Suppose D is a pseudoeffective divisor on $\text{Pic}^0(X)$ such that $D \cdot \gamma = 0$. Since $\text{Pic}^0(X)$ is an abelian variety, D is nef. By Proposition 1.11 the classes $[V^0(\mathcal{O}_{C_1}(x_1))^-]$ and $[V^0(\mathcal{O}_{C_2}(x_2))^-]$ are proportional to products of ample divisor classes on $\text{Pic}^0(C_1)$ and $\text{Pic}^0(C_2)$, respectively; it then follows from Lemma 1.8 that $D = 0$. \square

Combining the proof of Theorem B with the discussion from Remark 2.2 yields the following result for symmetric squares of curves.

Proposition 3.2. *Let C be a smooth projective curve of genus $g \geq 6$, and let $\mathcal{O}_C(1)$ be an ample and globally generated line bundle of degree $d < g - 3$ on C . If $X = C^{(2)}$ and $\mathcal{O}_X(1)$ is a line bundle on X whose pullback via the quotient map $\pi : C^2 \rightarrow X$ is isomorphic to $\mathcal{O}_C(1)^{\boxtimes 2}$, then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$. \square*

Proposition 3.3. *Let C_1 and C_2 be nonisomorphic smooth projective curves that do not admit nontrivial correspondences, and let G be a finite abelian group which acts freely on $C_1 \times C_2$ and faithfully on C_1 and C_2 . If $X := (C_1 \times C_2)/G$ and $\mathcal{O}_X(1)$ is an ample and globally generated line bundle on X , then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$.*

Proof. It suffices to exhibit $\eta \in \text{Pic}^0(X)$ such that $\mathcal{F} \otimes \eta$ is a GV-sheaf. Let $\pi : C_1 \times C_2 \rightarrow X$ be the quotient map; this is étale by our hypothesis on the action of G . By (1.1) of [Pa], we have that $\pi_* \mathcal{O}_{C_1 \times C_2} \cong \mathcal{O}_X \oplus \mathcal{G}$, where \mathcal{G} is a direct sum of nontrivial elements of $\text{Pic}^0(X)$.

By upper semicontinuity and Riemann-Roch, we may then choose $\eta \in \text{Pic}^0(X)$ such that $h^i(\mathcal{F}(j) \otimes \eta \otimes \xi) = h^i(\mathcal{F}(j))$ for $0 \leq i \leq 2, j \in \mathbb{Z}$, and any $\xi \in \text{Pic}^0(X)$ which is a direct summand of \mathcal{G} . It follows that $\mathcal{F}' := \mathcal{F} \otimes \eta$ is 1-regular with respect to $\mathcal{O}_X(1)$. We will now show that \mathcal{F}' (and therefore \mathcal{F}) is a GV-sheaf.

It is enough to show that $\pi^* \mathcal{F}'$ (which is torsion-free since π is étale) is 1-regular with respect to $\pi^* \mathcal{O}_X(1)$. Granting this for the moment, the absence of nontrivial correspondences on $C_1 \times C_2$ implies the existence of ample and globally generated line bundles $\mathcal{O}_{C_1}(1), \mathcal{O}_{C_2}(1)$ on C_1, C_2 resp. such that

$$(3.4) \quad \pi^* \mathcal{O}_X(1) \cong \mathcal{O}_{C_1}(1) \boxtimes \mathcal{O}_{C_2}(1)$$

Since $\pi^* \mathcal{F}'$ is assumed to be 1-regular with respect to $\mathcal{O}_{C_1}(1) \boxtimes \mathcal{O}_{C_2}(1)$, Proposition 3.1 implies that $\pi^* \mathcal{F}'$ is a GV-sheaf on $C_1 \times C_2$. For any $\eta' \in \text{Pic}^0(X)$, we have

$$(3.5) \quad H^i(\mathcal{F}' \otimes \eta') \subset H^i(\mathcal{F}' \otimes \eta' \otimes \pi_* \mathcal{O}_{C_1 \times C_2}) \cong H^i(\pi^* \mathcal{F}' \otimes \pi^* \eta')$$

Consequently $\pi^*(V^i(\mathcal{E}'(-1))) \subseteq V^i(\pi^* \mathcal{F}')$. Since the map $\pi^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(C_1 \times C_2)$ is finite onto its image, we have

$$(3.6) \quad \text{codim}(V^i(\mathcal{F}')) = \text{codim}(\pi^*(V^i(\mathcal{F}'))) \geq \text{codim}(V^i(\pi^* \mathcal{F}')) \geq i.$$

so that \mathcal{F}' is a GV-sheaf as claimed.

To check that $\pi^*\mathcal{F}'$ is 1-regular with respect to $\pi^*\mathcal{O}_X(1)$, note that for $0 \leq i \leq 2$ and $j \in \mathbb{Z}$ we have that

$$(3.7) \quad H^i(\pi^*\mathcal{F}'(j)) \cong H^i(\mathcal{F}'(j) \otimes \pi_*\mathcal{O}_{C_1 \times C_2}) \cong H^i(\mathcal{F}'(j)) \oplus H^i(\mathcal{F}'(j) \otimes \mathcal{G})$$

Due to the vanishings guaranteed by our definition of \mathcal{F}' , we have that $\pi^*\mathcal{F}'$ is 1-regular as desired. \square

4. RULED THREEFOLDS

The purpose of this final section is to prove the following result. Note that when the curve C has large genus, there are many possibilities for $\mathcal{O}_X(1)$ that are not adjunctions.

Proposition 4.1. *Let C be a smooth projective curve of genus $g \geq 1$, let \mathcal{V} be an ample and globally generated vector bundle of rank 3 on C , and let $X = \mathbb{P}(\mathcal{V})$ be the projectivization with structure map $\pi : X \rightarrow C$. If $\mathcal{O}_C(1)$ is a globally generated line bundle of degree $d \geq g$ on C and $\mathcal{O}_X(1) := \pi^*\mathcal{O}_C(1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$, then (\star) has an affirmative answer for $(X, \mathcal{O}_X(1))$.*

Proof. Let \mathcal{F} be a torsion-free coherent sheaf which is 1-regular with respect to $\mathcal{O}_X(1)$. It is already clear that $\text{codim}(V^1(\mathcal{F})) \geq 1$; we will show that $V^i(\mathcal{F}) = \emptyset$ for $i = 2, 3$.

Fix reduced divisors $D' \in |\mathcal{O}_C(1)|$ and $Z \in |\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)|$, and define $Y = \pi^{-1}(D')$. It is immediate that $H := Y + Z \in |\mathcal{O}_X(1)|$. Since $d \geq g$, we have from Riemann-Roch that $V^0(\mathcal{O}_C(1)) = \text{Pic}^0(C) \cong \text{Pic}^0(X)$. For each $\alpha \in \text{Pic}^0(C)$ we fix $D'_\alpha \in |\mathcal{O}_C(1) \otimes \alpha|$ and define $D_\alpha := \pi^{-1}(D'_\alpha) + Z$; this is algebraically equivalent to H . The 1-regularity hypothesis on \mathcal{F} implies that for each $\alpha \in \text{Pic}^0(C)$ and $i = 2, 3$ we have

$$(4.1) \quad H^i(\mathcal{F} \otimes \pi^*\alpha) = H^i(\mathcal{F} \otimes \mathcal{O}_X(D_\alpha - H)) \cong H^i(\mathcal{F}|_{D_\alpha} \otimes \pi^*\alpha)$$

This vanishes for dimension reasons when $i = 3$. It follows that $V^3(\mathcal{F}) = \emptyset$. For the remaining case $i = 2$, we consider the Mayer-Vietoris sequence

$$(4.2) \quad 0 \rightarrow (\mathcal{F} \otimes \pi^*\alpha)|_{D_\alpha} \rightarrow (\mathcal{F} \otimes \pi^*\alpha)|_{\pi^{-1}(D'_\alpha)} \oplus (\mathcal{F} \otimes \pi^*\alpha)|_Z \rightarrow (\mathcal{F} \otimes \pi^*\alpha)|_{\pi^{-1}(D'_\alpha) \cap Z} \rightarrow 0$$

Observe that $\pi^{-1}(D'_\alpha)$ is the disjoint union of d irreducible components, each of which is isomorphic to \mathbb{P}^2 . In particular, $\pi^{-1}(D'_\alpha) \cap Z$ is the disjoint union of d smooth irreducible curves, each of which is a line in a copy of \mathbb{P}^2 . Since $\mathcal{F}(1)$ is globally generated and torsion-free, it follows that the torsion-free summand of the restriction of \mathcal{F} to each irreducible component of $\pi^{-1}(D'_\alpha) \cap Z$ is a direct sum of line bundles of degree -1 or greater. Moreover, the restriction of $\pi^*\alpha$ to each component of $\pi^{-1}(D'_\alpha)$ is trivial, so $H^1((\mathcal{F} \otimes \pi^*\alpha)|_{\pi^{-1}(D'_\alpha) \cap Z}) = H^1(\mathcal{F}|_{\pi^{-1}(D'_\alpha) \cap Z}) = 0$. Consequently (4.2) implies that $H^2((\mathcal{F} \otimes \pi^*\alpha)|_{D_\alpha})$ injects into $H^2((\mathcal{F} \otimes \alpha)|_{Y_\alpha}) \oplus H^2((\mathcal{F} \otimes \pi^*\alpha)|_Z)$. We will have shown that $V^2(\mathcal{F}) = \emptyset$ once we verify that $H^2((\mathcal{F} \otimes \alpha)|_{Y_\alpha})$ and $H^2((\mathcal{F} \otimes \pi^*\alpha)|_Z)$ are both 0.

For the first summand, it is enough to check that $H^2(\mathcal{F}|_{Y_\alpha}) = 0$. We already know that $H^2(\mathcal{F}) = 0$, so $H^2(\mathcal{F}|_{Y_\alpha})$ injects into $H^3(\mathcal{F}(-Y_\alpha))$. Given that $2H$ is linearly equivalent to $(Y_\alpha + Z) + (Y_{-\alpha} + Z)$, we have that $H^3(\mathcal{F}(-Y_\alpha))$ is a quotient of $H^3(\mathcal{F}(-2))$, which is 0.

For the second summand, we observe that $H^2((\mathcal{F} \otimes \pi^*\alpha)|_Z) \cong H^1(R^1\pi_*(\mathcal{F}|_Z \otimes \alpha))$. Since $H^1(\mathcal{F}|_{Y_\alpha \cap Z}) = 0$ for all α by our previous argument, it follows from Grauert's theorem that $R^1\pi_{1*}(\mathcal{F}|_Z) = 0$, so $H^2((\mathcal{F} \otimes \pi^*\alpha)|_Z) = 0$ as desired. \square

REFERENCES

- [ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves, Volume I*, New York: Springer-Verlag, 1985.
- [Bea] A. Beauville, *Ulrich Bundles on Abelian Surfaces*, Proc. Amer. Math. Soc., to appear.
- [CP] A. Castorena and G.P. Pirola, *Some results on deformations of sections of vector bundles*, Collect. Math., to appear.
- [De] O. Debarre, *On coverings of simple abelian varieties*, Bull. Soc. math. France **134** (2006), p. 253-260.
- [DELV] O. Debarre, L. Ein, R. Lazarsfeld, and C. Voisin, *Pseudoeffective and nef classes on abelian varieties*, Compos. Math. **147** (2011), no. 6, p. 1793-1818.
- [ESW] D. Eisenbud, F.-O. Schreyer, and J. Weyman. *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc. **16** (2003), no. 3, p. 537-579.
- [FL] M. Fulger and B. Lehmann, *Positive cones of dual cycle classes*, Alg. Geom., to appear.
- [GL] M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), p. 389-407.
- [Ha] C. Hacon, *A derived category approach to generic vanishing*, J. Reine Angew. Math. **575** (2004)
- [Iz] E. Izadi, *Deforming curves in Jacobians to non-Jacobians I: Curves in $C^{(2)}$* , Geom. Dedicata **116** (2005), p. 87-109.
- [Laz] R. Lazarsfeld, *Positivity in Algebraic Geometry, Vol. 1*, Springer-Verlag, New York (2004).
- [LN] L. Lombardi and W. Niu, *Theta-Regularity of Curves and Brill-Noether Loci*, to appear.
- [MPP] M. Mendes Lopes, R. Pardini and G.P. Pirola, *Brill-Noether loci for divisors on irregular varieties*, J. Eur. Math. Soc. **16** (2014), p. 2033-2057.
- [Pa] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine. Angew. Math. **417** (1991), p. 191-213.
- [PP1] G. Pareschi and M. Popa, *Regularity on abelian varieties I*, J. Amer. Math. Soc. **16** (2003), p. 285-302.
- [PP2] G. Pareschi and M. Popa, *Castelnuovo theory and the geometric Schottky problem*, J. Reine. Angew. Math. **615** (2008), p. 25-44.
- [PP3] G. Pareschi and M. Popa, *Regularity on abelian varieties III: relationship with generic vanishing and applications*, in *Grassmannians, Moduli Spaces and Vector Bundles*, Clay Mathematics Proceedings **14** (2011), Amer. Math. Soc., Providence, RI, p. 141-167.
- [PP4] G. Pareschi and M. Popa, *GV-sheaves, Fourier-Mukai transform, and Generic Vanishing*, Amer. J. Math. **133** (2011), no.1, p. 235-271.

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