Vector bundles whose restriction to a linear section is Ulrich

Rajesh S. Kulkarni [*], Yusuf Mustopa [†], and Ian Shipman [‡]

July 30, 2015

Abstract

An Ulrich sheaf on an $n-$dimensional projective variety $X \subseteq \mathbb{P}^N$ is a normalized ACM sheaf which has the maximum possible number of global sections. Using a construction based on the representation theory of Roby-Clifford algebras, we prove that every normal ACM variety admits a reflexive sheaf whose restriction to a general 1-dimensional linear section is Ulrich; we call such sheaves $\delta-$Ulrich. In the case $n = 2$, where $\delta-$Ulrich sheaves satisfy the property that their direct image under a general, finite, linear projection to $\mathbb{P}^2$ is a semistable instanton bundle on $\mathbb{P}^2$, we show that some high Veronese embedding of X admits a $\delta-$Ulrich sheaf with a global section.

Introduction

The structure theory of ACM sheaves on a subvariety $X \subseteq \mathbb{P}^N$ is an important and actively studied area of algebraic geometry. Ulrich sheaves are the "nicest possible" ACM sheaves on X, since their associated Cohen-Macaulay module has the maximum possible number of generators, they are closed under extensions (they form an Abelian subcategory of $\mathrm{Coh}(X)$), and their Hilbert series is completely determined by their rank and $\deg(X)$. Moreover, they are all Gieseker-semistable.

Ulrich sheaves are known to exist on curves and Veronese varieties [ESW03] (and [Han99]), hypersurfaces [BHS88], complete intersections [BHU91], generic linear determinantal varieties [BHU87], Segre varieties [CMRPL12], rational normal scrolls [MR13], Grassmannians [CMR15], some flag varieties [CMR15, CHW], and generic K3 surfaces [AFO]. The question of whether every subvariety of projective space admits an Ulrich sheaf was first posed in [ESW03] and remains open. It was shown in [KMSb] that an affirmative answer is equivalent to the simultaneous solution of a large number of higher-rank Brill-Noether problems on nongeneric curves. In light of the fact that the varieties currently known to admit Ulrich sheaves are almost all ACM, a natural first step is to restrict the question to ACM varieties.

It is straightforward to check that if $\mathcal{E}$ is an Ulrich sheaf on X, then the restriction of $\mathcal{E}$ to a general linear section is Ulrich. The converse holds for linear sections of dimension 2 or greater (Lemma 3.1) but not linear sections of dimension 1 (e.g. Remark 3.6). In addition, Ulrich sheaves on 1-dimensional linear sections have very recently been used by Faenzi and Pons-Llopis to show that most ACM varieties are of wild representation type [FPL]. All this suggests a natural enlargement of the class of Ulrich sheaves whose existence problem may be more tractable.

[*]Michigan State University, East Lansing, Michigan. kulkarni@math.msu.edu

[†]Tufts University, Medford, Massachusetts. Yusuf.Mustopa@tufts.edu

[‡]University of Utah, Salt Lake City, Utah. ian.shipman@gmail.com

Definition. Let $\mathcal{E}$ be a reflexive sheaf on a polarized variety $(X, \mathcal{O}_X(1))$. We say that $\mathcal{E}$ is $\delta-$*Ulrich* if there exists a smooth 1-dimensional linear section Y of X such that the restriction $\mathcal{E}|_Y$ is an Ulrich sheaf on Y (that is, $h^0(\mathcal{E}|_Y(-1)) = 0$ and $h^0(\mathcal{E}|_Y) = \deg(Y) \cdot \mathrm{rank}(\mathcal{E})$).

Our main result (Theorem 2.7) is the following:

Theorem A. *Let $X \subseteq \mathbb{P}^N$ be a normal ACM variety. Then X admits a $\delta-$Ulrich sheaf.*

The $\delta-$Ulrich condition for a sheaf $\mathcal{F}$ on X can be rephrased as saying that if $\pi : X \to \mathbb{P}^n$ is a general finite linear projection, the direct image $\pi_*\mathcal{F}$ restricts to a trivial vector bundle on a general line $\ell \subseteq \mathbb{P}^n$, so to construct a $\delta-$Ulrich sheaf on X amounts to finding a reflexive sheaf $\mathcal{E}$ on $\mathbb{P}^n$ and a line $\ell \subseteq \mathbb{P}^n$ such that $\mathcal{E}$ is an $\pi_*\mathcal{O}_X-$modules and $\mathcal{E}|_\ell$ is a trivial vector bundle on ℓ. It suffices to carry this out this construction on an open subset of $\mathbb{P}^n$ whose complement is of codimension 2.

Lemma 2.4 implies that if $\pi : X \to \mathbb{P}^n$ is a finite linear projection, then there are open affine subsets $V_1, V_2 \subseteq \mathbb{P}^n$ and polynomials $p_i(z_i) \in \mathcal{O}_{V_i}[z_i]$ such that the complement of $V_1 \cup V_2$ is of codimension 2 and $\pi_*\mathcal{O}_X|_{V_i} \cong \mathcal{O}_{V_i}[z_i]/(p_i(z_i))$. Our strategy for proving Theorem A begins with constructing for $i = 1, 2$ a locally Cohen-Macaulay sheaf $\mathcal{E}_i$ on V_i which admits the structure of a $\pi_*\mathcal{O}_X|_{V_i}$-module. What allows us to do this is the notion of a *characteristic morphism* of (sheaves of) algebras. Such morphisms generalize algebra homomorphisms in the sense that they respect the Cayley-Hamilton theorem; see Section 1.1 for details, as well as [KMSa]. Although we are not aware of any earlier work on characteristic morphisms as such, we were inspired by the use of characteristic polynomials in [Pap00]. For similar ideas in the context of invariant theory, see [Pro87].

It is not obvious that $\mathcal{E}_1$ and $\mathcal{E}_2$ glue together to form a $\pi_*\mathcal{O}_X|_{V_1 \cup V_2}-$module. However, the special characteristic morphism we construct in Proposition 2.5 ensures that the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to a general line $\ell \subseteq V_1 \cup V_2$ glue together to form an Ulrich sheaf for the restriction $\pi^{-1}(\ell) \to \ell$ of π. The $\delta-$Ulrich sheaf we produce is an algebraization of a sheaf on the formal neighborhood of ℓ which comes from gluing completions of $\mathcal{E}_1$ and $\mathcal{E}_2$ along this neighborhood (Lemma 2.6 and Theorem 2.7).

Even though it is not used explicitly, the central concept underlying the proof of Proposition 2.5 is that of the Roby-Clifford algebra R_F of a degree-d homogeneous form F over a field $\mathbf{k}$. This was introduced by Roby in [Rob69], and it directly generalizes the classical Clifford algebra of a quadratic form, as R_F satisfies a similar, higher-degree universal property (see Remark 1.2). It is shown in [VdB87] that Ulrich sheaves on the cyclic covering hypersurface $\{w^d = F\}$ correspond to finite-dimensional R_F-modules, and a more refined correspondence involving the natural $\mathbb{Z}/d\mathbb{Z}$-grading on R_F is used in [BHS88] to construct Ulrich sheaves on hypersurfaces. The latter construction uses the *$\mathbb{Z}/d\mathbb{Z}$-graded tensor product* of modules over Roby-Clifford algebras (see Section 1.2) to construct an Ulrich sheaf over the zero locus of the "generic homogeneous form of degree d which is a sum of s monomials." Our proof of Proposition 2.5 uses $\mathbb{Z}/d\mathbb{Z}$-graded tensor products to extend an algebraic structure (the characteristic morphism) from the line $\ell \subset \mathbb{P}^n$ to all of $\mathbb{P}^n$.

We can say more about $\delta-$Ulrich sheaves when X is a normal ACM surface. It is immediate from the definition that $\delta-$Ulrich sheaves on normal ACM surfaces are locally Cohen-Macaulay, a necessary condition for being Ulrich. When $X = \mathbb{P}^2$, the sheaves which are $\delta-$Ulrich with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$ are semistable instanton sheaves in the sense of [Jar06], so in general, $\delta-$Ulrich sheaves on a surface have the property that their direct image under a finite linear projection is a semistable instanton sheaf (Proposition 4.1). We show that the intermediate cohomology module $H^1_*(\mathcal{E})$ satisfies the Weak Lefschetz property (Proposition 4.11); moreover, the maximum value of the Hilbert function of $H^1_*(\mathcal{E})$ is $h^1(\mathcal{E}(-1))$.

A substantial difference between Ulrich and $\delta-$Ulrich sheaves is that the former are globally generated, while the latter need not have any global sections at all (compare Remark 3.6). However, a $\delta-$Ulrich sheaf $\mathcal{E}$ on X is Ulrich if and only if it has $\deg(X) \cdot \mathrm{rk}(\mathcal{E})$ global sections (see Proposition

3.2). If we replace $\mathcal{O}_X(1)$ by a potentially high twist, we have enough control on the cohomology to obtain the following result.

Theorem B. *If $X \subseteq \mathbb{P}^N$ is a smooth ACM surface, there exists $k > 0$ such X admits a $\delta-$Ulrich sheaf with respect to $\mathcal{O}_X(k)$ possessing a global section.*

This theorem follows from a more precise statement. If $\mathcal{E}$ is a $\delta-$Ulrich sheaf on X, consider the quantity

$$\alpha(\mathcal{E}) = h^0(\mathcal{E})/\deg(X)\mathrm{rk}(\mathcal{E})$$

Our earlier observation can be rephrased as saying that $\mathcal{E}$ is Ulrich if and only if $\alpha(\mathcal{E}) = 1$. Theorem B is proved by exhibiting a sequence of sheaves $\{\mathcal{E}_m\}_m$ where $\mathcal{E}_m$ is δ-Ulrich with respect to $\mathcal{O}_X(2^m)$ and such that $\lim_{m\to\infty} \alpha(\mathcal{E}_m) = 1$.

Acknowledgments

I.S. was partially supported during the preparation of this paper by National Science Foundation award DMS-1204733. R. K. was partially supported by the National Science Foundation awards DMS-1004306 and DMS-1305377.

Notation and Conventions

Our base field $\mathbf{k}$ is algebraically closed of characteristic zero. All open subsets are Zariski-open. If R is a ring we use the notation $R\{t_1, \dots, t_n\}$ for the free R-module with basis $t_1, \dots, t_n$.

1 Preliminaries

In this section, we collect the algebraic prerequisites for the proof of Theorem A. Throughout, R denotes a commutative $\mathbf{k}-$algebra and A denotes a commutative $R-$algebra which is free of rank $d \geq 2$ as an $R-$module.

1.1 Roby Modules and Characteristic Morphisms

Definition 1.1. Let M, W be free $R-$modules and let $F \in \mathrm{Sym}^\bullet_R(M^\vee)$ be a homogeneous form of degree $e \geq 2$. An $R-$module morphism $\phi : M \to \mathrm{End}_R(W)$ is an $F-$Roby module if for all $m \in M$ we have

$$\phi(m)^e = F(m) \cdot \mathrm{id}_W$$

where $F(m)$ is the image of $m^{\otimes e}$ under the symmetric map $M^{\otimes e} \to R$ associated to F. If, in addition, W is a $\mathbb{Z}/e\mathbb{Z}$-graded $R-$module and $\phi(m)$ is a degree-1 endomorphism for $0 \neq m \in M$, we say that ϕ is a *graded* $F-$Roby module.

Remark 1.2. The terminology can be explained as follows. If ϕ is an $F-$Roby module, the induced $R-$algebra morphism $T^\bullet_R(M) \to \mathrm{End}_R(W)$ annihilates $\{\phi(m)^e - F(m) : m \in M\}$, and therefore descends to a morphism $R_F \to \mathrm{End}_R(W)$, where

$$R_F := T^\bullet_R(M)/\langle \phi(m)^e - F(m) : m \in M\rangle$$

is the Roby-Clifford algebra of F (see [Rob69]). Conversely, given an $R-$algebra morphism $R_F \to \mathrm{End}_R(W)$, we recover an $F-$Roby module by composing with the natural injection $M \hookrightarrow R_F$.

Example 1.3. We recall a construction from [Chi78]. Let $M = R\{x_1, \ldots, x_n\}$ and suppose that $y_1, \ldots, y_n$ is the dual basis of $M^\vee$. Consider a monomial $F = y_{i_1} y_{i_2} \ldots y_{i_e} \in \mathrm{Sym}^e_R(M^\vee)$ and put $W = R\{w_1, \ldots, w_e\}$. Then there is a natural, $\mathbb{Z}/e\mathbb{Z}-$graded $F-$Roby module $\phi : M \to \mathrm{End}_R(W)$ given by

$$\phi(x_i)(w_j) = \begin{cases} w_{j+1} & i = i_j, \\ 0 & \text{otherwise,} \end{cases}$$

where the indices on the elements $w_1, \ldots, w_e$ are taken modulo e and $\deg(w_i) = i$.

Definition 1.4. The *characteristic polynomial* of A is

$$\chi_A(t, a) := \det(tI - \rho_A(a)) = \sum_{j=0}^{d} (-1)^j \mathrm{tr}(\wedge^j \rho_A(a)) \cdot t^{d-j}$$

where $\rho_A : A \to \mathrm{End}_R(A)$ is the regular representation of A.

Observe that $\chi_A(t, a)$ is a degree-d element of $\mathrm{Sym}^\bullet_R(A^\vee) \otimes_R R[t] \cong \mathrm{Sym}^\bullet_R(A^\vee \oplus R\{t\})$. Also, if B is an $R-$algebra, then for any $a \in A$ and $b \in B$ we have that $\chi_A(b, a)$ is a well-defined element of B.

Example 1.5. Consider the $R-$algebra $A = R^{\times d}$. We identify $R^{\times d} = R\{e_1, \ldots, e_d\}$ where $\{e_i\}$ is the standard basis of idempotents. Under the regular representation we have $\rho_A(a_1, \ldots, a_d) = \mathrm{diag}(a_1, \ldots, a_d)$ and therefore $\chi_A(t, a_1, \ldots, a_d) = (t - a_1) \cdots (t - a_d)$. It folows that

$$\chi_A(t) = (t - x_1) \cdots (t - x_d)$$

where $x_1, \ldots, x_d$ is the dual basis to $e_1, \ldots, e_d$.

We record the following elementary properties, which will be used in the sequel.

Lemma 1.6.

1. *If B is a commutative $R-$algebra which is free of finite rank as an $R-$module, then χ_A is taken to $\chi_{A \otimes_R B}$ under the natural map*

$$\mathrm{Sym}^\bullet_R(A^\vee)[t] \to \mathrm{Sym}^\bullet_B((A \otimes_R B)^\vee)[t]$$

induced by the base-change map $A^\vee \to (A \otimes_R B)^\vee = \mathrm{Hom}_B(A \otimes_R B, B)$.

2. *If $B \to C$ is an embedding of $R-$algebras, both free of the same finite rank then χ_B is the image of χ_C under the natural morphism*

$$\mathrm{Sym}^\bullet_R(C^\vee)[t] \to \mathrm{Sym}^\bullet_R(B^\vee)[t].$$

□

If $\phi : A \to B$ is a morphism of $R-$algebras, the Cayley-Hamilton theorem implies that

$$\chi_A(\phi(a), a) = \phi(\chi_A(a, a)) = 0$$

for all $a \in A$. The more general notion that follows is a key ingredient in our construction of $\delta-$Ulrich sheaves.

Definition 1.7. If B is an $R-$algebra, an $R-$module morphism $\phi : A \to B$ is a *characteristic morphism* if $\chi_A(\phi(a), a) = 0$ for all $a \in A$.

Remark 1.8. The notion of a characteristic morphism is strictly more general than that of an $R-$algebra morphism. If $A = R\{e_1, e_2\}$ is the $R-$algebra generated by the orthogonal idempotents e_1 and e_2, then for any $a, b \in R$ satisfying $a + b \neq 0$, the map $\phi : A \to \mathrm{Mat}_2(R)$ defined by

$$\phi(e_1) = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \ \ \phi(e_2) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

is a characteristic morphism, but not an $R-$algebra morphism.

We now turn to the sheaf-theoretic formulations of these concepts. For the remainder of this subsection, Y denotes a smooth quasi-projective variety, $\mathcal{A}$ denotes a sheaf of $\mathcal{O}_Y-$algebras which is locally free of rank $d \geq 2$, and W denotes a finite-dimensional $k-$vector space. For a sheaf $\mathcal{F}$ on Y, we denote the stalk of $\mathcal{F}$ at a point $y \in Y$ by $\mathcal{F}_y$.

Definition 1.9. If $\mathcal{B}$ is a coherent sheaf of $\mathcal{O}_Y-$algebras, a $\mathcal{O}_Y-$linear morphism $\phi : \mathcal{A} \to \mathcal{B}$ is a *characteristic morphism* if for each $y \in Y$, the $\mathcal{O}_{Y,y}-$module morphism $\phi_y : \mathcal{A}_y \to \mathcal{B}_y$ is a characteristic morphism.

The following observation will be used later.

Lemma 1.10. *$\phi : \mathcal{A} \to \mathcal{B}$ is a characteristic morphism if and only if the induced $k(Y)-$linear map $\phi_{k(Y)} : \mathcal{A}(Y) \to \mathcal{B}(Y)$ is a characteristic morphism.* □

If $\rho_A : \mathcal{A} \to \mathcal{E}nd(\mathcal{A})$ is the regular representation of $\mathcal{A}$, then since $\mathrm{tr}(\wedge^j \rho_{\mathcal{A}})$ is a global section of $\mathrm{Sym}^j(\mathcal{A}^\vee)$ for each j, there exists a global characteristic polynomial $\chi_{\mathcal{A}} \in H^0(\mathrm{Sym}^\bullet(\mathcal{A}^\vee \oplus \mathcal{O}_Y\{t\}))$.

Definition 1.11. An $\mathcal{O}_Y-$linear morphism $\psi : \mathcal{A} \oplus \mathcal{O}_Y\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ is a *$\chi_{\mathcal{A}}-$Roby module* if for each $y \in Y$, the $\mathcal{O}_{Y,y}-$module morphism $\psi_y : \mathcal{A}_y \oplus \mathcal{O}_{Y,y}\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_{Y,y})$ is a $\chi_{\mathcal{A}}-$Roby module in the sense that for all $a \in \mathcal{A}_y$ and all $r \in \mathcal{O}_{Y,y}$ we have

$$\psi(a, rT)^d = \chi_{\mathcal{A}}(a, r) \cdot \mathrm{Id}.$$

If W is $\mathbb{Z}/d\mathbb{Z}-$graded and $\psi(a, rT)$ is a degree-1 endomorphism for all local sections a, r then ψ is a *graded $\chi_{\mathcal{A}}-$Roby module.*

If ψ is a graded $\chi_{\mathcal{A}}$-Roby module as above, then $\psi(T)$ is globally defined, and $\psi(T)^d = \mathrm{Id}$ since $\chi_{\mathcal{A}}$ is monic in t. In particular, $\psi(T)$ is invertible in $\mathrm{End}(W \otimes \mathcal{O}_Y)$.

Lemma 1.12. *Let $\psi : \mathcal{A} \oplus \mathcal{O}_Y\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ be a graded Roby $\chi_{\mathcal{A}}-$module. Then the morphism $C_\psi : \mathcal{A} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ defined by the composition*

$$\mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathcal{O}_Y\{T\} \xrightarrow{-\psi} \mathrm{End}(W \otimes \mathcal{O}_Y) \xrightarrow{\cdot \psi(T)^{-1}} \mathrm{End}(W \otimes \mathcal{O}_Y)$$

is a characteristic morphism.

Proof. By Lemma 1.10 it suffices to consider a field extension $K/\mathbf{k}$ and a $d-$dimensional commutative $K-$algebra A in place of $\mathcal{O}_Y$ and $\mathcal{A}$, respectively. Put $W_K = W \otimes_{\mathbf{k}} K$. Let $\chi_A = \chi_A(t)$ be the characteristic polynomial of A, and let $\psi : A \oplus K\{T\} \to \mathrm{End}_K(W_K)$ be a graded χ_A-Roby module. Then ψ corresponds to an element $\psi^\vee$ of $\mathrm{End}_K(W_K) \otimes (A^\vee \oplus K\{t\})$ whose $d-$th power

$$(\psi^\vee)^d \in \mathrm{End}_K(W_K) \otimes \mathrm{Sym}^d(A^\vee \oplus K\{t\}) \cong \mathrm{Hom}(W_K, W_K \otimes \mathrm{Sym}^d(A^\vee \oplus K\{t\}))$$

is equal to $1_W \otimes \chi_A$.

Consider the graded $S = \mathrm{Sym}^\bullet(A^\vee)[t,w]/(w^d - \chi_A)$-module $M = W_K \otimes_K \mathrm{Sym}^\bullet(A^\vee)[t]$ on which w acts by $\psi^\vee$ (and $A^\vee, t$ act in the obvious way). Now, M is a graded maximal Cohen-Macaulay S-module, generated in degree zero (a graded Ulrich module in fact). So if $R \subset S$ is any standard-graded polynomial subring of S over which S is finite and flat, M will be graded-free over it and generated in degree zero. In particular we can consider $\mathrm{Sym}^\bullet(A^\vee)[w] \subset S$. Then the map

$$W_K \otimes_K \mathrm{Sym}^\bullet(A^\vee)[w] \to M$$

is an isomorphism. We aim to compute the action of t in terms of the action of w and $A^\vee$. We can write

$$\psi^\vee = \psi_0^\vee + \psi(T) \otimes t, \quad \psi_0^\vee \in \mathrm{End}_K(W_K) \otimes A^\vee.$$

So if $m \in M$ we have

$$wm = \psi_0^\vee m + t\psi(T)m.$$

As we observed earlier, $\psi(T)$ is invertible. We deduce that

$$tm = w\psi(T)^{-1}m - \psi_0^\vee \psi(T)^{-1}m.$$

Reduce M modulo w to obtain a module over the ring $\mathrm{Sym}^\bullet(A^\vee)[t]/(\chi_A(t))$ which is graded-free over $\mathrm{Sym}^\bullet(A^\vee)$ and generated in degree zero. Now, the action of t on this module is given by $-\psi_0^\vee \psi(T)^{-1}$. Since $\chi_A(t)$ is zero in this ring, we see that the map $A \to \mathrm{End}_K(W_K)$ corresponding to $-\psi_0^\vee \psi(T)^{-1}$ is a characteristic morphism. This map is C_ψ so we see that C_ψ is a characteristic morphism. $\square$

Example 1.13. Again consider $A = R^{\times d} = R\{e_1, \dots, e_d\}$. From Example 1.5 we see that $\chi_A(t) = \prod_{i=1}^d (t - x_i)$ where $\{x_i\}$ is the dual basis to $\{e_i\}$. There is a natural graded χ_A-Roby module $\phi : A \oplus R\{T\} \to \mathrm{End}_R(R\{w_1, \dots, w_d\})$ defined by

$$\phi(T)(w_i) = w_{i+1}, \quad \phi(e_i)(w_j) = \begin{cases} -w_{j+1} & i = j+1, \\ 0 & i \neq j+1, \end{cases}$$

where the indices on the w_i are taken modulo d and $\deg(w_i) = i$. Since

$$\phi(rT + \sum_{i=1}^r a_i e_i)(w_j) = (r - a_{j+1})w_{j+1}$$

we see by iteration that ϕ is indeed a χ_A-Roby module. Let us compute C_ϕ. We have

$$C_\phi(e_i)(w_j) = -\phi(e_i)(\phi(T)^{-1}(w_j)) = -\phi(e_i)(w_{j-1}) = \delta_j^i w_j.$$

So, better than simply being a characteristic morphism we see that C_ϕ is an algebra morphism, equipping W with the structure of a free $R^{\times d}$-module. However, note that the construction of this module implicitly relied on a cyclic ordering on the idempotents $e_\bullet \in A$.

Remark 1.14. We note that the formation of C_ϕ is functorial. More precisely, suppose that $\phi : \mathcal{A} \oplus \mathcal{O}\{T\} \to \mathrm{End}(W) \otimes \mathcal{O}$ is a χ-Roby morphism. If $W' \subset W$ is an invariant subspace in the sense that for any local section $a + rT$ of $\mathcal{A} \oplus \mathcal{O}\{T\}$, the action of $\phi(a, rT)$ on $W \otimes \mathcal{O}$ sends $W' \otimes \mathcal{O}$ into itself, then the action of $\mathcal{A}$ via C_ϕ will also send $W' \otimes \mathcal{O}$ into itself. This equips $W' \otimes \mathcal{O}$ and $W/W' \otimes \mathcal{O}$ with the structures of χ-Roby modules and characteristic modules, respectively.

1.2 $\mathbb{Z}/d\mathbb{Z}-$Graded Tensor Products

The following notion, which was first applied to the study of Roby modules in [Chi78], is required for the proof of Proposition 2.5. The proof is essentially that of Theorem 3.1 in [BHS88].

Proposition-Definition 1.15. *Let M be a free $R-$module and $F_1, F_2 \in \mathrm{Sym}_R^d(M^\vee)$ homogeneous forms. Suppose that for each $i = 1, 2$ we have a graded F_i-Roby module $\phi_i : M \to \mathrm{End}_R(W_i)$, where W_1, W_2 are $\mathbb{Z}/d\mathbb{Z}-$graded $R-$modules. Then the morphism $\phi : M \to \mathrm{End}_R(W_1 \otimes W_2)$ defined by*

$$\phi(m)(w_1 \otimes w_2) = \phi_1(m)(w_1) \otimes w_2 + \xi^{\deg(w_1)} w_1 \otimes \phi_2(m)(w_2)$$

is a graded $F_1 + F_2-$Roby module, where $W_1 \otimes W_2$ is graded by $\deg(w_1 \otimes w_2) = \deg(w_1) + \deg(w_2)$ for homogeneous elements $w_i \in W_i$. We denote this morphism by $\phi = \phi_1 \widehat{\otimes}_\xi \phi_2$.

2 Construction of δ-Ulrich Sheaves

We now take up the proof of Theorem A in earnest. As our first step, we use Lemma 1.12 to produce a type of enhanced Ulrich sheaf for any finite covering of $\mathbb{P}^1$.

Lemma 2.1. *Let C be a smooth curve and let $f : C \to \mathbb{P}^1$ be a morphism of degree $d \geq 2$ Then there exists a graded $\chi_{f_*\mathcal{O}_C}$-Roby module ψ whose associated characteristic morphism C_ψ is an $f_*\mathcal{O}_C-$module morphism.*

Proof. Let $K(C)/K(\mathbb{P}^1)$ be the field extension corresponding to f. Since the extension is separated, it has the form $K(C) \cong K(\mathbb{P}^1)[z]/(p(z))$ for some polynomial $p(z)$. Let L be the splitting field of $p(z)$ and $g : D \to \mathbb{P}^1$ the map of curves corresponding to $L/K(\mathbb{P}^1)$. Then $C \times_{\mathbb{P}^1} D$ has d components, each of which is isomorphic to D. So we have a diagram

$$\begin{tikzcd} \sqcup_{i=1}^d D \arrow[dr, "\eta"] & & \\ & C \times_{\mathbb{P}^1} D \arrow[r] \arrow[d] & D \arrow[d, "g"] \\ & C \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

where η is the normalization of $C \times_{\mathbb{P}^1} D$. Let $\mathcal{A} = f_*\mathcal{O}_C$, $\mathcal{B} = g_*\mathcal{O}_D$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$. Since C and D are reduced curves, they are locally CM, so $\mathcal{A}$ and $\mathcal{B}$ are locally free as $\mathcal{O}-$modules; in particular, $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$ is locally free as a $\mathcal{B}-$module. Also, η is induced by a $\mathcal{B}-$module morphism $\widetilde{\eta} : \mathcal{A} \otimes_{\mathcal{O}} \mathcal{B} \to \mathbf{k}^{\times d} \otimes_{\mathbf{k}} \mathcal{B}$.

Let $\chi(t) \in \mathrm{Sym}_{\mathcal{O}}^\bullet(\mathcal{A}^\vee)[t]$, $\widetilde{\chi}(t) \in \mathrm{Sym}_{\mathcal{B}}^\bullet((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$, and $\widetilde{\chi}_s(t) \in \mathrm{Sym}_{\mathcal{B}}^\bullet((\mathcal{B}^\vee \otimes_{\mathbf{k}} \mathbf{k}^{\times d})^\vee)[t]$ be the characteristic polynomials of $\mathcal{A}$ over $\mathcal{O}$, $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$ over $\mathcal{B}$, and $\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}$ over $\mathcal{B}$ respectively. Then (according to Lemma 1.6) under the natural maps

$$\mathrm{Sym}_{\mathcal{B}}^\bullet((\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d})^\vee)[t] \to \mathrm{Sym}_{\mathcal{B}}^\bullet((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$$
$$\mathrm{Sym}_{\mathcal{O}}^\bullet(\mathcal{A}^\vee)[t] \to \mathrm{Sym}_{\mathcal{B}}^\bullet((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$$

we see that $\widetilde{\chi}_s(t)$ maps to $\widetilde{\chi}(t)$ and $\chi(t)$ maps to $\widetilde{\chi}(t)$. This means that if a and r are local sections of $\mathcal{A}$ and $\mathcal{O}$, respectively, then $\widetilde{\chi}_s(a, rt) = \chi(a, rt)$.

As in Example 1.13, there is a natural graded $\widetilde{\chi}_s(t)$-Roby module $\phi : (\mathbf{k}^{\times d} \otimes_{\mathbf{k}} \mathcal{B}) \oplus \mathcal{B}\{T\} \to \mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d})$ defined by

$$\phi(T)(e_j) = e_{j+1}, \quad \psi(e_i)(e_j) = \begin{cases} -e_{j+1} & i = j+1 \\ 0 & i \neq j+1 \end{cases}$$

where the indices of the standard idempotents e_i are taken modulo d. Clearly, ϕ is a Roby module for the characteristic polynomial $\widetilde{\chi}$ for $\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}$ over $\mathcal{B}$. Moreover, one can verify that C_ψ is an algebra morphism.

By [ESW03], there is an Ulrich sheaf $\mathcal{E}$ for D over $\mathbb{P}^1$, which we view as a $\mathcal{B}$-module on $\mathbb{P}^1$. Note that there is an algebra morphism

$$\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}_{\mathcal{O}}(\mathcal{E} \otimes \mathbf{k}^{\times d})$$

defined by tensoring a map with $\mathcal{E}$ over $\mathcal{B}$. We can then obtain a map

$$\widetilde{\phi} : \mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d} \oplus \mathcal{B} \cdot T \to \mathrm{End}(\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}) \to \mathrm{End}_{\mathcal{O}}(\mathcal{E} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}) = \mathrm{End}(W \otimes \mathbf{k}^{\times d} \otimes \mathcal{O})$$

where we choose a trivialization $\mathcal{E} \cong W \otimes \mathcal{O}$ as $\mathcal{O}$-modules. Now this induces a χ-Roby module structure on $W \otimes \mathbf{k}^{\times d} \otimes \mathcal{O}$. Let $\psi : \mathcal{A} \oplus \mathcal{O}\{T\} \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$ be the restriction of $\widetilde{\phi}$ to $\mathcal{A} \oplus \mathcal{O}\{T\} \subset \mathcal{B} \otimes \mathbf{k}^{\times d} \oplus \mathcal{B}\{T\}$, where $\mathcal{A} \to \mathcal{B} \otimes \mathbf{k}^{\times d}$ is the map $\mathcal{A} \otimes 1 \to \mathcal{A} \otimes \mathcal{B} \xrightarrow{\widetilde{\eta}} \mathcal{B} \otimes \mathbf{k}^{\times d}$.

Suppose that a, r are local sections of $\mathcal{A}$ and $\mathcal{O}$ respectively. Then we know that $\phi(a,r)^d = \widetilde{\chi}_s(a,r) \cdot \mathrm{id} \in \mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d})$. Now $\widetilde{\chi}_s(a,r) = \chi(a,r) \in \mathcal{O}$. Since the morphism $\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$ is $\mathcal{O}$-linear, ψ is a χ-Roby module. Finally, we must check that C_ψ is a morphism. Now, $C_\phi = R_{\phi(T)^{-1}}\phi$ is a morphism, where $R_{\phi(T)^{-1}}$ is right multiplication by $\phi(T)^{-1}$. Composing C_ϕ with the map $\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$, we obtain $R_{\widetilde{\phi}(T)^{-1}}\widetilde{\phi}$ and this is still a morphism. Now, if we restrict this morphism to $\mathcal{A}$ we obtain $C_\psi = R_{\psi(T)^{-1}}\psi$ since $\psi(T) = \widetilde{\phi}(T)$ and ψ is the restriction of $\widetilde{\phi}$ to $\mathcal{A}$. ☐

Definition 2.2. Let W be a vector space, and let $F^\bullet$ be an increasing filtration on W. A *filtered pseudomorphism* $\phi : \mathcal{A} \to \mathrm{End}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ is a characteristic morphism satisfying the following properties:

(i) The image of ϕ is contained in the algebra $\mathrm{End}_{F^\bullet}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ of endomorphisms preserving $F^\bullet$.

(ii) The induced map $\phi_{F^\bullet} : \mathcal{A} \to \Pi_i \mathrm{End}(F^{i+1}W/F^iW \otimes \mathcal{O}_{\mathbb{P}^n})$ is an $\mathcal{O}_{\mathbb{P}^n}$-algebra morphism.

Our $\delta-$Ulrich sheaf will come from a characteristic morphism that restricts to a filtered pseudomorphism on a 1-dimensional linear section.

Definition 2.3. A coherent sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathbb{P}^n}-$algebras is said to be *monogenic* on an open set $U \subseteq \mathbb{P}^n$ if there exists a monic polynomial $p[z] \in \mathcal{O}_U[z]$ and an isomorphism $\mathcal{A}|_U \cong \mathcal{O}_U[z]/\langle p(z) \rangle$ of $\mathcal{O}_U-$algebras.

Lemma 2.4. *Let $X \subseteq \mathbb{P}^N$ be a subvariety of dimension n which is regular in codimension 1, where $2 \leq n \leq N-2$. Then for a general finite linear projection $\pi : X \to \mathbb{P}^n$, there are affine open sets $U_1, U_2 \subseteq \mathbb{P}^n$ satisfying the following conditions:*

(i) *$\pi_*\mathcal{O}_X$ is monogenic on U_1 and U_2.*

(ii) *$U_1 \cup U_2$ contains a line ℓ such that $X_\ell := \pi^{-1}(\ell)$ is smooth and contained in the regular locus X^{reg}.*

Proof. Consider the space of triples $P \subset X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1) \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n)$ defined as the closure of

$$P^o = \{(x, \Lambda', \Lambda) : x \in X^{reg},\, x \in \Lambda,\, \Lambda' \subset \Lambda,\, \dim(T_xX \cap T_x\Lambda) > 1\}.$$

Let $P' \subset X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ be the image of P under projection. For a general (x, Λ') in P', we have $x \notin \Lambda'$ and therefore Λ is the projective span of Λ' and x. So $\dim(P) = \dim(P')$. Let Q be the image of P in $X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n)$. Then $P \to Q$ is generically a projective space bundle whose fibers have dimension $N-n$. So $\dim(P) = \dim(Q) + N - n$. We will compute the dimension of Q, using the projection to X. Let $x \in X^{reg}$. Then the fiber of Q over x is birationally isomorphic to the set of pairs

$$\{(\alpha, \Lambda) : \alpha \in \mathrm{Gr}(T_x X, 2), x \in \Lambda \in \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n), \alpha \subset T_x\Lambda\}.$$

This set of pairs is a $\mathrm{Gr}(N-2, N-n-2)-$bundle over $\mathrm{Gr}(n,2)$. So it has dimension $2(n-2) + n(N-n-2)$. Hence we see that $\dim(Q) = n + 2(n-2) + n(N-n-2)$. Finally we deduce that

$$\dim(P') = N + 2(n-2) + n(N-n-2).$$

Now if $P' \to \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ is dominant then a general fiber has dimension

$$N + 2(n-2) + n(N-n-2) - (N-n)(n+1) = n-4.$$

If $P' \to \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ is not dominant, then a general $(N-n-1)-$plane Λ would have the property that for any $x \in X^{reg}$, $\dim(T_x\Lambda_x \cap T_x X) \leq 1$, where Λ_x is the projective span of Λ and x. If $P' \to \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ is dominant, then for a general $(N-n-1)-$plane Λ, we have $\dim(T_x\Lambda_x \cap T_x X) \leq 1$ away from a subset of X^{reg} of codimension at least 4. In either case, for a general $(N-n-1)-$plane Λ, $\Lambda \cap X = \varnothing$, Λ_x is tranverse to X at a general point, and off a locus of codimension two, $T_x\Lambda_x \cap T_x X$ is at most one dimensional. Fixing such a general Λ, let us denote by Z the union of the bad locus and the singular locus of X.

Let $\pi : X \to \mathbb{P}^n$ be the finite projection associated to Λ. Say that $p \in \mathbb{P}^n \setminus \pi(Z)$ and consider the fiber $\pi^{-1}(p)$. For each $x \in \pi^{-1}(p)$ we see that since $T_x\Lambda_x \cap T_x X$ is at most one dimensional, the cotangent space to $\pi^{-1}(p)$ at x is at most one dimensional. Hence $\pi^{-1}(q)$ is monogenic.

Consider an affine open set $U \subset \mathbb{P}^n \setminus \pi(Z)$. Write $\mathcal{A} = \pi_*\mathcal{O}_X$, viewed as a locally free sheaf of $\mathcal{O}_{\mathbb{P}^n}$ algebras. Let $u \in U$ be some point and let $z \in \mathcal{A}(U)$ be an element such that $z|_u$ is a generator for $\mathcal{A}|_u$. Then there is a polynomial $p(z)$ (the characteristic polynomial of z) such that the map $\mathcal{O}_U[z]/\langle p(z)\rangle \to \mathcal{A}|_U$ is an isomorphism away from a divisor $D \subset U$. Put $U_1 = U \setminus D$. Note that U_1 is affine and $\mathcal{A}$ is monogenic on U_1.

Let $\ell \subset \mathbb{P}^n$ be a line which avoids Z, has nonempty intersection with U_1, and is such that $\pi^{-1}(\ell)$ is smooth. Let $y_1, \ldots, y_r$ be the points on $\ell \cap (\mathbb{P}^n \setminus U_1)$. By construction, the fiber of $\mathcal{A}$ at each y_i is monogenic. Let $V \subset \mathbb{P}^n \setminus \pi(Z)$ be an affine open that contains all of the y_i. For each i, let $z_i \in \mathcal{A}|_{y_i}$ be a generator for the algebra. Since $\mathcal{A}(V) \to \prod_{i=1}^r \mathcal{A}|_{y_i}$ is surjective, there is an element $z \in \mathcal{A}(V)$ whose restriction to y_i is z_i. Now as before there is polynomial $q(z)$ such that the map $\mathcal{O}_V[z]/\langle q(z)\rangle \to \mathcal{A}|_V$ is an isomorphism away from a divison $D' \subset V$. By construction $y_1, \ldots, y_r \notin D'$. Put $U_2 = V \setminus D'$. Then $\mathcal{A}$ is monogenic on U_2 and moreover $\ell \subset U_1 \cup U_2$. ☐

Proposition 2.5. *Let $X \subseteq \mathbb{P}^N$ be a normal ACM variety of dimension $n \geq 2$, and let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. If $\ell \subseteq \mathbb{P}^n$ is a line, there exists a filtered vector space W and a characteristic morphism $\phi : \mathcal{A} \to \mathrm{End}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ such that $\phi|_\ell$ is a filtered pseudomorphism.*

Proof. Let $x, y, z_2, \ldots, z_n$ be a coordinate system on $\mathbb{P}^n$ such that $\ell = V(z_2, \ldots, z_n)$. It is convenient to work with graded rings instead of schemes. So let us view $\mathbb{P}^n = \mathrm{Proj}(R)$ where $R = \mathbf{k}[x, y, z_2, \ldots, z_n]$ and $X = \mathrm{Proj}(S)$ where S is a graded Cohen-Macaulay R-algebra. Given that S is a free $R-$module, we fix a homogeneous basis $1 = \gamma_1, \ldots, \gamma_d$ for S as an R-module. Note that $\deg(\gamma_i) > 0$ for $i > 1$. Moreover $\ell = \mathrm{Proj}(\mathbf{k}[x,y])$. Let $\overline{S} = S/(z_2, \ldots, z_n)S$ and write $\chi(t)$ and $\chi_\ell(t)$ for the characteristic polynomials of S over R and $\overline{S}$ over $\mathbf{k}[x,y]$, respectively.

As in Lemma 2.1 we can find a graded $\chi_\ell(t)$-Roby module

$$\phi_\ell : \overline{S} \oplus \mathbf{k}[x,y] \cdot T \to \mathrm{End}(W) \otimes \mathbf{k}[x,y].$$

Such that C_{ϕ_ℓ} is a morphism. Recall that ϕ_ℓ must have the property that

$$\phi_\ell(\alpha_1\gamma_1 + \ldots + \alpha_d\gamma_d + \tau T)^d = \chi_\ell(\alpha_1\gamma_1 + \ldots + \alpha_d\gamma_d + \tau t) \cdot \mathrm{id}_W$$

where $\alpha_\bullet, \tau \in \mathbf{k}[x, y]$. Let us view $\mathbf{k}[x, y] \subset \mathbf{k}[x, y, z_2, \ldots, z_n]$ as a subring. Then put $\phi_0 = \phi_\ell \otimes_{\mathbf{k}[x,y]} R$. Write $\chi_0(t)$ for $\chi_\ell(t)$ viewed as an element of $\mathrm{Sym}^\bullet_R(S^\vee)[t]$. Then ϕ_0 is a graded $\chi_0(t)$-Roby module.

Now we note that $\chi(t) - \chi_0(t) \in (z_\bullet)\,\mathrm{Sym}^\bullet_R(S^\vee)[t]$. So let us write

$$\chi(t) = \chi_0(t) + \sum_{i=0}^{d-1}\sum_{j=1}^{n_i} t^i(c_{i,j,1}\Gamma_{k(i,j,1)})(c_{i,j,2}\Gamma_{k(i,j,2)})\cdots(c_{i,j,d-i}\Gamma_{k(i,j,d-i)})$$

where $\Gamma_1, \ldots, \Gamma_d$ are the variables dual to the basis $\gamma_1, \ldots, \gamma_d$ and $c_{i,j,s} \in R$ has degree equal to $\deg(\gamma_{k(i,j,s)})$. Now put $m_{i,j} = t^i(c_{i,j,1}\Gamma_{k(i,j,1)}) \ldots (c_{i,j,d-i}\Gamma_{k(i,j,d-i)})$.

We recall the construction of Example 1.3. Define a map $\phi_{m_{i,j}} : S \oplus R\{T\} \to \mathrm{End}(R \otimes \mathbf{k}^d)$ by

$$\phi_{m_{i,j}}(\gamma_p)(\epsilon_r) = c_{i,j,r-i}\delta^p_{k(i,j,r-i)}\epsilon_{r+1}, \quad (i < r \leq d), \quad \phi_{m_{i,j}}(T)(\epsilon_r) = \begin{cases} \epsilon_{r+1} & 1 \leq r \leq i, \\ 0 & i < r \leq d \end{cases}$$

where $\epsilon_\bullet$ are the standard basis vectors of $\mathbf{k}^d$ and addition in the subscripts are modulo d.

Now consider $\phi = \phi_0 \widehat{\otimes}_\xi \phi_{m_{0,1}} \widehat{\otimes}_\xi \cdots \widehat{\otimes}_\xi \phi_{m_{d,n_d}}$ which is a $\chi_{\mathcal{A}}-$Roby action on $\widetilde{W} = (\mathbf{k}^d)^{\otimes n_0 + \cdots + n_{d-1}} \otimes W \otimes R$. By Proposition 1.15, this is a χ-Roby module. For each i, j at least one of the $c_{i,j,r}$ must be in the ideal $(z_\bullet)$. After possibly reindexing, we may assume that $c_{i,j,d-i} \in (z_\bullet)$.

Consider the filtration $F^i\mathbf{k}^d = \mathbf{k}\{\epsilon_i, \ldots, \epsilon_d\}$ on each "monomial" Roby module. Then upon restriction to ℓ, the Roby-action of $\overline{S} \oplus \mathbf{k}[x, y]\{T\}$ on $\mathbf{k}^{\times d} \otimes \mathbf{k}[x, y]$ via $\phi_{m_{i,j}}$ preserves $F^\bullet$. Moreover, if we put $F^{d+1} = 0$, we have that for each i,

$$(\overline{S} \oplus \mathbf{k}[x, y]\{T\}) \cdot F^i \subset F^{i+1}$$

Let us equip $\widetilde{W}$ with the filtration $\widehat{F}^\bullet$ which is the tensor product of the filtrations above on its monomial factors and the trivial filtration on $W \otimes R$. Then the action of $\overline{S} \oplus \mathbf{k}[x, y]\{T\}$ preserves $\widehat{F}^\bullet$. The formation of the $\mathbb{Z}/d\mathbb{Z}-$graded tensor product is bi-functorial on Roby modules. Since each $m_{i,j}$ vanishes in $\mathbf{k}[x, y, t]$, we see that the minimal subquotients of $\widehat{F}^\bullet$ are simply the $\mathbb{Z}/d\mathbb{Z}-$graded tensor product of ϕ_ℓ with a number of copies of the rank-one 0-Roby module correpsonding to the zero map $\overline{S} \oplus \mathbf{k}[x, y]\{T\} \to \mathrm{End}(\mathbf{k}[x, y])$. Therefore the minimal subquotients are isomorphic to ϕ_ℓ.

Finally, the formation of C_ϕ is functorial. So the action of $\overline{S}$ via C_ϕ preserves the tensor product filtration. Since the associated graded parts are isomorphic to ϕ_ℓ and C_{ϕ_ℓ} is a morphism, we find that $C_\phi|_\ell$ is a filtered pseudo-morphism. □

Lemma 2.6. *Let $\mathcal{A}$ be an ACM sheaf of algebras on $\mathbb{P}^N$ and $\ell \subset \mathbb{P}^N$ a line. Let $\widehat{\mathcal{A}}$ be the formal completion of $\mathcal{A}$ along ℓ and let $\widehat{\mathcal{E}}$ be a coherent sheaf of $\widehat{\mathcal{A}}$-modules. Then there is a coherent sheaf $\mathcal{E}$ of $\mathcal{A}$-modules whose completion is isomorphic to $\widehat{\mathcal{E}}$.*

Proof. Let $\mathcal{I}$ be the ideal defining ℓ. Consider the sheaf of algebras $\mathcal{S} = \oplus_{m \geq 0}\mathcal{I}^m/\mathcal{I}^{m+1}$ and the coherent graded $\mathcal{S}$ module $\mathcal{F} = \oplus_{m \geq 0}\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}}$. Note that $\mathcal{S}(1)$ is ample on $\mathrm{Spec}(\mathcal{S})$, where $\mathcal{S}(1)$ is the pullback of $\mathcal{O}_\ell(1)$ under the natural map $\mathrm{Spec}(\mathcal{S}) \to \ell$. It follows that for some $n_0 \gg 0$, $\mathrm{H}^1(\mathcal{F}(n_0)) = 0$. Observe that

$$\mathrm{H}^1(\mathrm{Spec}(\mathcal{S}), \mathcal{F}(n_0)) = \oplus_m \mathrm{H}^1(\ell, (\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)),$$

since $\operatorname{Spec}(\mathcal{S}) \to \ell$ is affine. Therefore, for each m,

$$\mathrm{H}^1((\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)) = 0.$$

It follows that the maps

$$\mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)) \to \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}^{m}\widehat{\mathcal{E}})(n_0))$$

are surjective. Therefore the map $\mathrm{H}^0(\widehat{\mathcal{E}}(n_0)) \to \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}})(n_0))$ is surjective. If $V \subset \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}})(n_0))$ is a finite dimensional space of sections such that

$$V \otimes \mathcal{A}(-n_0) \to \widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}}$$

is surjective and $V' \subset \mathrm{H}^0(\widehat{\mathcal{E}}(n_0))$ is a lift then

$$V' \otimes \widehat{\mathcal{A}}(-n_0) \to \widehat{\mathcal{E}}$$

is also surjective. Indeed, the support of the cokernel is empty. Iterating this arugment we obtain a presentation

$$W \otimes \widehat{\mathcal{A}}(-n_1) \xrightarrow{\hat{\alpha}} V' \otimes \widehat{\mathcal{A}}(-n_0) \to \widehat{\mathcal{E}} \to 0.$$

Now consider the map $\mathrm{H}^0(\mathcal{A}(k)) \to \mathrm{H}^0(\widehat{\mathcal{A}}(k))$. We wish to show that it is surjective. Since $\mathcal{A}$ is dissocié, it suffices to show that the maps $\mathrm{H}^0(\mathcal{O}(k)) \to \mathrm{H}^0(\widehat{\mathcal{O}}(k))$ are surjective for all $k \gg 0$. If $m > k$ then $\mathrm{H}^0(\mathcal{O}(k)) \to \mathrm{H}^0((\mathcal{O}/\mathcal{I}^m)(k))$ is an isomorphism. Hence $\mathrm{H}^0(\mathcal{A}(k)) \to \mathrm{H}^0(\widehat{\mathcal{A}}(k))$ is an isomorphism. Therefore there is a morphism

$$\alpha : W \otimes \mathcal{A}(-n_1) \to V' \otimes \mathcal{A}(-n_0)$$

whose completion is $\hat{\alpha}$. Thus we may take $\mathcal{E} = \operatorname{coker}(\alpha)$. □

The next result completes the proof of Theorem A.

Theorem 2.7. *Under the hypothesis of Proposition 2.5, there exists a 1-dimensional linear section $C \subseteq X$ and a reflexive sheaf $\mathcal{E}$ on X such that $\mathcal{E}|_C$ is Ulrich.*

Proof. Let $p : X \to \mathbb{P}^n$ and V_1, V_2 be as in Lemma 2.4 and let $\ell \subset V_1 \cup V_2$ be a line. Next, let $(W, F^\bullet)$ be a filtered vector space and $\phi : \mathcal{A} = p_*\mathcal{O}_X \to \operatorname{End}(W \otimes \mathcal{O})$ be as in Lemma 2.5 with respect to the line ℓ. Let $z_i \in \mathcal{A}(V_i)$ be an algebra generator for $\mathcal{A}$ over V_i. Consider the algebra map ϕ_i defined by the diagram

$$\mathcal{A}_i = \mathcal{A}_{V_i} \xleftarrow{\cong} \mathcal{O}_{V_i}[z_i]/(p_i(z_i)) \longrightarrow \operatorname{End}(W \otimes \mathcal{O}_{V_i})$$

where the second map is unique algebra homomorphism which sends z_i to $\phi(z_i)$. Write $\mathcal{E}_i$ for $W \otimes \mathcal{O}_{V_i}$ with the $\mathcal{A}_{V_i}$-module structure coming from ϕ_i. Note that $\mathcal{E}_i$ is maximal Cohen-Macaulay over V_i and therefore locally free on $V_i \setminus p(\operatorname{sing}(X))$. In particular, $\mathcal{E}_i$ is locally free in a neighborhood of $\ell \cap V_i$.

Let $F^\bullet$ be the filtration on W. By assumption, the pseudomorphism $\phi : \mathcal{A}|_\ell \to \operatorname{End}(W \otimes \mathcal{O}_\ell)$ induces a morphism on $\mathcal{A}|_\ell \to \prod \operatorname{End}(F^{i+1}W/F^iW \otimes \mathcal{O}_\ell)$. Since X_ℓ is smooth the $\mathcal{A}|_\ell$ module structure on $F^{i+1}W/F^iW \otimes \mathcal{O}_\ell$ is locally free. Now, $V_1 \cap V_2 \cap \ell$ is affine. This means that the filtration $F^\bullet\mathcal{E}_{ij}$ has projective subquotients. So there is an isomorphism $\operatorname{gr}_{F^\bullet} \mathcal{E}_{ij} \to \mathcal{E}_{ij}$ which is compatible with the filtration when $\operatorname{gr}_{F^\bullet} \mathcal{E}_{ij}$ is filtered by $F^k = \oplus_{k' \le k} F^{k'}\mathcal{E}_{ij}/F^{k'-1}\mathcal{E}_{ij}$ and which induces the identity on subquotients. Using these isomorphisms we produce a filtered $\mathcal{A}_{12}$-module isomorphism

$$\psi_\ell : \mathcal{E}_{12}|_\ell \to \mathcal{E}_{21}|_\ell$$

which induces the same isomorphism on associated graded modules as the identification $\mathcal{E}_{12} = W \otimes \mathcal{O}_{\ell \cap V_{12}} = \mathcal{E}_{21}$. Let $\mathcal{F}$ be the vector bundle on ℓ obtained by gluing $\mathcal{E}_1|_\ell$ to $\mathcal{E}_2|_\ell$ along ψ_ℓ. Now since ψ_ℓ is filtered, $\mathcal{F}$ is filtered. By construction, the subquotients of $\mathcal{F}$ for this filtration are the same as the subquotients for $W \otimes \mathcal{O}_\ell$. Hence the associated graded of $\mathcal{F}$ is trivial. It follows that $\mathcal{F}$ is itself a trivial vector bundle.

Let $\widehat{U}$ be the formal neighborhood of ℓ in $\mathbb{P}^n$. Let $j : \widehat{U} \to \mathbb{P}^n$ and put $\widehat{V}_i = j^{-1}(V_i)$ and $\widehat{\mathcal{A}} = j^*\mathcal{A}$. Write $\widehat{\mathcal{E}}_i = j^*\mathcal{E}_i$. Since $\mathcal{E}_i$ is a locally free $\mathcal{A}_i$ module on a neighborhood of $\ell \cap V_i$ we see that $\widehat{\mathcal{E}}_i$ is a locally free $\widehat{\mathcal{A}}$-module. Since $\mathbb{P}^n$ is separated, $V_{12} = V_1 \cap V_2$ is affine. Hence $\widehat{V}_{12} = \widehat{V}_1 \cap \widehat{V}_2$ is an affine formal scheme. Hence the $\widehat{\mathcal{E}}_i$ are projective $\widehat{\mathcal{A}}_{V_i}$-modules. Therefore the isomorphism ψ_ℓ lifts to an isomorphism

$$\psi : \widehat{\mathcal{E}}_{12} \to \widehat{\mathcal{E}}_{21}$$

of $\widehat{\mathcal{A}}_{12}$-modules. The isomorphism ψ gives gluing data for gluing $\widehat{\mathcal{E}}_1$ to $\widehat{\mathcal{E}}_2$. Let $\widehat{\mathcal{E}}$ be $\widehat{\mathcal{A}}$-module obtained by gluing $\widehat{\mathcal{E}}_1$ to $\widehat{\mathcal{E}}_2$ along ψ. By Lemma 2.6, there is a sheaf $\mathcal{E}$ of $\widehat{\mathcal{A}}$-modules whose restriction to $\widehat{U}$ is $\widehat{\mathcal{E}}$. Since $\widehat{\mathcal{E}}$ is locally free, $\mathcal{E}$ is locally free in a neighborhood of ℓ. So replacing $\mathcal{E}^{\vee\vee}$ is also isomorphic to $\widehat{\mathcal{E}}$ on $\widehat{U}$. Hence $\mathcal{E}^{\vee\vee}$ is the desired sheaf. □

3 Generalities on $\delta-$Ulrich Sheaves

In this section, $X \subseteq \mathbb{P}^N$ is a normal ACM variety of degree d and dimension $n \geq 2$.

Lemma 3.1. *Suppose that $\mathcal{E}$ is a locally CM sheaf on X whose restriction to a linear section Y of dimension at least 2 is Ulrich. Then $\mathcal{E}$ is Ulrich.*

Proof. Let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. Since $\mathcal{E}$ is a locally CM sheaf on X, the direct image $\pi_*\mathcal{E}$ is a locally CM sheaf on a smooth variety, and is therefore locally free. Replacing $\mathcal{E}$ by $\pi_*\mathcal{E}$ if necessary, we can assume without loss of generality that $\mathcal{E}$ is locally free and $X = \mathbb{P}^n$. We will show that $\mathcal{E}$ is trivial.

By induction on dimension we may assume that $\dim(Y) = n - 1$ so that Y is a hyperplane. Our hypothesis on $\mathcal{E}$ amounts to $\mathcal{E}|_Y$ being a trivial bundle. To show that $\mathcal{E}$ is trivial, it is enough to check that $h^0(\mathcal{E}) = \mathrm{rk}(\mathcal{E})$. (See Proposition 3.2.) We will do this by showing that restriction map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_Y)$ is surjective, since $h^0(\mathcal{E}|_Y) = \mathrm{rk}(\mathcal{E})$.

For each positive integer j, let jY the $(j-1)-$st order thickening of Y. We claim that for all $m \geq 1$, the restriction map $H^0(\mathcal{E}|_{(m+1)Y}) \to H^0(\mathcal{E}|_{mY})$ is an isomorphism. Grant this for the time being. If we fix $m_0 \gg 0$ for which $H^1(\mathcal{E}(-m_0)) = 0$, then the restriction map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_{m_0 Y})$ is surjective, and the claim yields the desired surjectivity of the map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_Y)$.

Turning to the proof of this claim, an obvious snake lemma argument gives the following exact sequence for each $m \geq 1$:

$$0 \to \mathcal{E}|_Y(-m) \to \mathcal{E}|_{(m+1)Y} \to \mathcal{E}|_{mY} \to 0.$$

Since $\mathcal{E}|_Y$ is trivial and $\dim(Y) > 1$, $h^0(\mathcal{E}|_Y(-m)) = h^1(\mathcal{E}|_Y(-m)) = 0$, so the map $H^0(\mathcal{E}|_{(m+1)Y}) \to H^0(\mathcal{E}|_{mY})$ is an isomorphism.

□

If $\mathcal{E}$ is a $\delta-$Ulrich sheaf on X then for a general 1-dimensional linear section $Y \subset X$, $\mathcal{E}|_Y$ is Ulrich. Indeed, if we consider a finite linear projection $\pi : X \to \mathbb{P}^n$ then $\pi_*\mathcal{E}$ is a reflexive sheaf whose restriction to a given line is trivial. Since trivial sheaves on $\mathbb{P}^1$ are rigid, the restriction of $\pi_*\mathcal{E}$ to nearby lines is also trivial. We also point out that if X_H is a general hyperplane section of X then $\mathcal{E}|_{X_H}$ is δ-Ulrich on X_H.

Proposition 3.2. *Let $\mathcal{E}$ be a $\delta-$Ulrich sheaf of rank r on X. Then the following are equivalent.*

(i) *$\mathcal{E}$ is Ulrich.*

(ii) *$h^0(\mathcal{E}) = dr$.*

Proof. It is clear that (i) implies (ii). So assume $h^0(\mathcal{E}) = dr$. Replacing $\mathcal{E}$ by its direct image under a finite linear projection if necessary, we can assume that $X = \mathbb{P}^N$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1)$ (in particular, $d = 1$) without loss of generality. In this case the Ulrich condition on $\mathcal{E}$ is equivalent to $\mathcal{E} \cong \mathcal{O}^r_{\mathbb{P}^n}$ and the $\delta-$Ulrich condition on $\mathcal{E}$ is equivalent to $\mathcal{E}|_\ell \cong \mathcal{O}^r_\ell$ for some line $\ell \subseteq \mathbb{P}^n$. If $h^0(\mathcal{E}) = \mathrm{rk}(\mathcal{E})$ then since $h^0(\mathcal{E} \otimes \mathcal{I}_\ell) = 0$ we find that the evaluation map $ev : \mathrm{H}^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ restricts to the evaluation map $\mathrm{H}^0(\mathcal{E}|_\ell) \otimes \mathcal{O}_\ell \to \mathcal{E}|_\ell$, which is an isomorphism. Let $\mathcal{F}$ be the cokernel of ev. Then the support of $\mathcal{F}$ has codimension at least two. We compute

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}^r) \cong \mathrm{Ext}^{n-1}(\mathcal{O}^r, \mathcal{F}(-(n+1))) \cong \mathrm{H}^{n-1}(\mathcal{F}(-(n+1)))^r = 0.$$

However, $\mathcal{E}$ is reflexive and in particular torsion free. Thus ev is an isomorphism. □

We shall now consider stability properties of $\delta-$Ulrich bundles.

Lemma 3.3. *Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Then $\mathcal{E}$ is $\mu-$semistable, and $\omega_X \otimes \mathcal{E}^\vee(n+1)$ is also δ-Ulrich.*

Proof. Let $\mathcal{F}$ be a torsion-free quotient of $\mathcal{E}$, and let $Y \subset X$ be a smooth 1-dimensional linear section such that $\mathcal{E}|_Y$ is Ulrich; we may also assume Y avoids the singular loci of $\mathcal{E}$ and $\mathcal{F}$. Then $\mathcal{F}|_Y$ is a torsion-free quotient of the semistable bundle $\mathcal{E}|_Y$; consequently $\mu(\mathcal{E}) = \mu(\mathcal{E}|_Y) \leq \mu(\mathcal{F}_Y) = \mu(\mathcal{F})$.

The second part of the statement follows from the adjunction formula and the fact that if C is a curve embedded in projective space by $\mathcal{O}_C(1)$ and $\mathcal{E}'$ is an Ulrich bundle on C, then $\omega_C \otimes \mathcal{E}'^\vee(2)$ is also Ulrich. □

Lemma 3.4. *If $\mathcal{E}$ is a δ-Ulrich sheaf on X which is strictly $\mu-$semistable, then there exists a $\mu-$stable subsheaf $\mathcal{E}' \subset \mathcal{E}$ which is $\delta-$Ulrich.*

Proof. Let $\mathcal{E}' \subset \mathcal{E}$ be the maximal destabilizing subsheaf of $\mathcal{E}$. We will show that $\mathcal{E}'$ is $\delta-$Ulrich. Let $Y \subseteq X$ be a general 1-dimensional linear section of $\mathcal{E}$ which avoids the singular loci of $\mathcal{E}'$ and $\mathcal{E}$. Then $\mathcal{E}'|_Y(-1)$ is a subsheaf of $\mathcal{E}|_Y(-1)$. Since the latter is an Ulrich sheaf, we have that $h^0(\mathcal{E}|_Y(-1)) = 0$, and it follows that $h^0(\mathcal{E}'|_Y(-1)) = 0$ as well. The slope of $\mathcal{E}'|_Y(-1)$ is equal to that of $\mathcal{E}|_Y(-1)$, so Riemann-Roch implies that $h^0(\mathcal{E}'|_Y(-1)) = h^1(\mathcal{E}'|_Y(-1)) = 0$; therefore $\mathcal{E}'|_Y$ is Ulrich. □

Lemma 3.5. *Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Then for all $k \geq 1$, we have $h^0(\mathcal{E}(-k)) = 0$.*

Proof. We proceed by induction on $\dim(X)$. Let $X_H \subset X$ be a general hyperplane section. Then for each $k \geq 1$ we have the exact sequence

$$0 \to \mathcal{E}(-k-1) \to \mathcal{E}(-k) \to \mathcal{E}|_{X_H}(-k) \to 0$$

Since negative twists of an Ulrich sheaf have no global sections, our inductive hypothesis implies $h^0(\mathcal{E}_{X_H}(-k)) = 0$; it follows that $h^0(\mathcal{E}(-k)) = h^0(\mathcal{E}(-1))$ for all $k \geq 1$. We need only exhibit some $k' \geq 1$ such that $h^0(\mathcal{E}(-k')) = 0$. Since $\mathcal{E}$ and all its twists are μ-semistable by Lemma 3.3, any positive $k' > \mu(\mathcal{E})$ will do. □

Remark 3.6. We exhibit for each $n \geq 2$ a smooth ACM variety of dimension n admitting $\delta-$Ulrich sheaves which are not Ulrich. Consider the Segre variety $X := \mathbb{P}^1 \times \mathbb{P}^{n-1} \subseteq \mathbb{P}^{2n-1}$, and let H be the hyperplane class of X. Recall that X is cut out in $\mathbb{P}^{2n+1}$ by the maximal minors of the generic $2 \times n$ matrix of linear forms. It follows from Proposition 2.8 of [BHU87] that the degeneracy locus $D \subseteq X$ of the first row of this matrix is a divisor whose associated line bundle $\mathcal{O}_X(D)$ is an Ulrich line bundle on X. The general 1-dimensional linear section $X' \subset X$ is a rational normal curve of degree n, so if $\mathcal{L} \in \mathrm{Pic}(X)$ satisfies $H^{n-1} \cdot \mathcal{L} = 0$, the restriction $\mathcal{L}|_{X'}$ is the trivial bundle; in particular $\mathcal{L}(D)$ is $\delta-$Ulrich. Since the set $(H^{n-1})^{\perp}$ of all such $\mathcal{L}$ is a corank-1 subgroup of $\mathrm{Pic}(X)$, we can choose $\mathcal{L} \in (H^{n-1})^{\perp}$ such that $\mathcal{L}(D)$ lies outside the effective cone of X, e.g. satisfies $H^0(\mathcal{L}(D)) = 0$. In this case $\mathcal{L}(D)$ is not Ulrich.

4 The surface case

Throughout this section we consider a normal surface X with a very ample line bundle $\mathcal{O}_X(1)$. We assume that X has a δ-Ulrich sheaf $\mathcal{E}$, but not necessarily that X is ACM.

4.1 Relation to Instanton Bundles

Proposition 4.1. *Let $\mathcal{E}$ be a $\delta-$Ulrich sheaf of rank r on X, and let $\pi : X \to \mathbb{P}^2$ be a finite linear projection. Then $\pi_*\mathcal{E}$ is $\mu-$semistable, and it is an instanton bundle on $\mathbb{P}^2$, i.e. the cohomology of a monad of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus m} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus rd+2m} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus m} \to 0 \tag{1}$$

where $d = \deg(X)$ and $m = h^1(\mathcal{E}(-1))$.

Proof. If $\mathcal{E}$ is $\delta-$Ulrich, then $\pi_*\mathcal{E}$ is a reflexive, and thus locally free, sheaf on $\mathbb{P}^2$. Since the restriction of $\pi_*\mathcal{E}$ to a general line is trivial, $\pi_*\mathcal{E}$ is $\mu-$semistable of degree 0, and given that $h^1(\mathcal{E}(-1)) = h^1(\pi_*\mathcal{E}(-1))$, Theorem 17 of [Jar06] implies our result. □

The following statement can be obtained from a short elementary argument, but it seems appropriately stated as a consequence of Proposition 4.1.

Corollary 4.2. *A $\delta-$Ulrich sheaf $\mathcal{E}$ on X is Ulrich if and only if $H^1(\mathcal{E}(-1)) = 0$.* □

At this point it is natural to ask if, given a $\delta-$Ulrich sheaf $\mathcal{E}$ on X, there is a $\delta-$Ulrich sheaf $\mathcal{E}'$ on X with $h^1(\mathcal{E}'(-1)) < h^1(\mathcal{E}(-1))$; an affirmative answer combined with Theorem A would imply that every normal ACM surface admits an Ulrich sheaf. The next result shows that it is enough to consider stable $\delta-$Ulrich bundles.

Lemma 4.3. *Let X be a smooth ACM surface, and let $\mathcal{E}$ be a δ-Ulrich sheaf on X which is strictly μ-semistable with $h^1(\mathcal{E}(-1)) = m$. Then X admits a locally free δ-Ulrich sheaf $\mathcal{E}'$ with $\mathrm{rk}(\mathcal{E}') < \mathrm{rk}(\mathcal{E})$ and $h^1(\mathcal{E}'(-1)) \leq \frac{m}{2}$.*

Proof. A μ-Jordan-Hölder filtration of $\mathcal{E}$ yields an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{E}(-1) \to \mathcal{G}(-1) \to 0$$

where $\mathcal{F}$ and $\mathcal{G}$ are both δ-Ulrich sheaves and $\mathcal{F}$ is both $\mu-$stable and locally free. Since $h^2(\mathcal{F}(-1)) = h^0(\omega_X \otimes \mathcal{F}^\vee(1))$, and $\omega_X \otimes \mathcal{F}^\vee(3)$ is δ-Ulrich by Lemma 3.3, we have from Lemma 3.5 that $h^2(\mathcal{F}(-1)) = 0$. Another application of this Lemma implies that $h^0(\mathcal{G}(-1)) = 0$. We may then conclude that $\min\{h^1(\mathcal{F}(-1)), h^1(\mathcal{G}(-1))\} \leq \frac{m}{2}$, and the result follows by taking the reflexive hull of $\mathcal{G}$ if necessary. □

Remark 4.4. Suppose that $X \subset \mathbb{P}^N$ is an ACM variety of dimension n and let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. Jardim [Jar06] defines a notion of instanton sheaves on $\mathbb{P}^n$ for any $n > 1$. However, only in the case $n = 2$ is the direct image $\pi_*\mathcal{E}$ of a $\delta-$Ulrich sheaf on X clearly an instanton sheaf. For $n > 2$, an instanton sheaf must satisfy additional cohomology-vanishing which does not follow from having trivial restriction to a line. Our construction does not appear to allow for any control over the cohomology of $\delta-$Ulrich sheaves.

4.2 Proof of Theorem B

For the next two Lemmas, we consider a δ-Ulrich sheaf $\mathcal{E}$ on $\mathbb{P}^2$ for the canonical polarization $\mathcal{O}_{\mathbb{P}^2}(1)$. In general, it is difficult to understand how an abstract δ-Ulrich sheaf will restrict to a curve in $\mathbb{P}^2$. However, we can say something when the curve is a general smooth conic.

Lemma 4.5. *Let $C \subset \mathbb{P}^2$ be a general smooth conic. Then $\mathcal{E}|_C$ is trivial.*

Proof. Any smooth conic is isomorphic to $\mathbb{P}^1$, so it is enough to show that $\mathcal{E}|_C$ is of degree 0 and semistable when C is a general element of $|\mathcal{O}_{\mathbb{P}^2}(2)|$. Since the restriction of $\mathcal{E}$ to a general line is a trivial bundle, it follows that $\det(\mathcal{E})$ is trivial. Consequently the restriction of $\mathcal{E}$ to any plane curve has degree 0. We now turn to semistability. Consider the universal plane conic

$$\mathcal{C} := \{(p, C) \in \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)| : p \in C\}$$

with its associated projections $p_1 : \mathcal{C} \to \mathbb{P}^2, p_2 : \mathcal{C} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$. Our goal amounts to showing that the restriction of $p_1^*\mathcal{E}$ to the general fiber of p_2 is semistable. Given that this is an open condition on the fibers of p_2 (e.g. Proposition 2.3.1 in [HL10]) it is enough to check the semistability of $\mathcal{E}|_{C_0}$ when $C_0 = L \cup L'$ for distinct lines $L, L' \subseteq \mathbb{P}^2$ satisfying the property that $\mathcal{E}|_L$ and $\mathcal{E}|_{L'}$ are trivial. If we twist the Mayer-Vietoris sequence

$$0 \to \mathcal{O}_{C_0} \to \mathcal{O}_L \oplus \mathcal{O}_{L'} \to \mathcal{O}_{L \cap L'} \to 0$$

by $\mathcal{E}$ and take cohomology, we see that the induced difference map $H^0(\mathcal{E}|_L) \oplus H^0(\mathcal{E}|_{L'}) \to H^0(\mathcal{E}|_{L\cap L'})$ is surjective. Therefore $\mathcal{E}|_{C_0}$ is locally free of rank $\mathrm{rk}(\mathcal{E})$ with $\mathrm{rk}(\mathcal{E})$ global sections, i.e. $\mathcal{E}|_{C_0} \cong \mathcal{O}_{C_0}^{\oplus \mathrm{rk}(\mathcal{E})}$. In particular, $\mathcal{E}|_{C_0}$ is semistable. $\square$

Lemma 4.6. *Let $\mathcal{F}$ be an $\mathcal{O}(2)$-Ulrich sheaf on $\mathbb{P}^2$. Then $\mathcal{E} \otimes \mathcal{F}$ is δ-Ulrich for $\mathcal{O}(2)$ and we have*

$$\chi(\mathcal{E} \otimes \mathcal{F}) = \mathrm{rk}(\mathcal{F})\left(\chi(\mathcal{E}) + 3\mathrm{rk}(\mathcal{E})\right).$$

Proof. Since the restriction of $\mathcal{E}$ and $\mathcal{F}$ to a general conic are trivial and Ulrich, respectively, and Ulrich sheaves are stable under taking direct sums, we see that the restriction of $\mathcal{E} \otimes \mathcal{F}$ to a general conic is Ulrich. Hence $\mathcal{E} \otimes \mathcal{F}$ is $\delta-$Ulrich for $\mathcal{O}(2)$.

Since $\mathcal{E}$ is $\delta-$Ulrich for $\mathcal{O}_{\mathbb{P}^2}(1)$, Proposition 4.1 implies that it is the cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^m \to \mathcal{O}_{\mathbb{P}^2}^{\mathrm{rk}(\mathcal{E})+2m} \to \mathcal{O}_{\mathbb{P}^2}(1)^m \to 0$$

where $m = h^1(\mathcal{E}(-1))$. Twisting by $\mathcal{F}$, we have that if $\ell \subseteq \mathbb{P}^2$ is a line, then

$$\begin{aligned}
\chi(\mathcal{E} \otimes \mathcal{F}) &= (\mathrm{rk}(\mathcal{E}) + 2m) \cdot \chi(\mathcal{F}) - m \cdot (\chi(\mathcal{F}(-1)) + \chi(\mathcal{F}(1))) \\
&= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) + m \cdot ((\chi(\mathcal{F}) - \chi(\mathcal{F}(-1))) - (\chi(\mathcal{F}(1)) - \chi(\mathcal{F}))) \\
&= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) + m \cdot (\chi(\mathcal{F}|_\ell) - \chi(\mathcal{F}(1)|_\ell)) \\
&= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) - m \cdot \mathrm{rk}(\mathcal{F})
\end{aligned}$$

We have from Riemann-Roch that $\chi(\mathcal{E}) = \mathrm{ch}_2(\mathcal{E}) + \mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) - m$; also, the fact that $\mathcal{F}$ is Ulrich with respect to $\mathcal{O}(2)$ implies that $\chi(\mathcal{F}) = 4\mathrm{rk}(\mathcal{F})$. Summarizing, we have that

$$\chi(\mathcal{E} \otimes \mathcal{F}) = 4\mathrm{rk}(\mathcal{E}) \cdot \mathrm{rk}(\mathcal{F}) + (\chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})) \cdot \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F})\left(\chi(\mathcal{E}) + 3\mathrm{rk}(\mathcal{E})\right).$$

□

One way to explain the previous Lemma is that Ulrich sheaves on $\mathbb{P}^2$ for $\mathcal{O}(2)$ are slightly positive. (The main example of an $\mathcal{O}(2)$-Ulrich sheaf is the tangent bundle $T\mathbb{P}^2$.) So tensoring with such a sheaf should enlarge the space of sections while decreasing the the higher cohomology. The Lemma makes this intuition precise and the next Theorem uses this idea to produce δ-Ulrich sheaves with sections (after changing the polarization).

Theorem 4.7. *Assume that $(X, \mathcal{O}_X(1))$ admits a δ-Ulrich sheaf. Then there exists a sequence $\mathcal{E}_m$ of sheaves on X such that $\mathcal{E}_m$ is δ-Ulrich for $\mathcal{O}_X(2^m)$ and*

$$\lim_{m\to\infty} \alpha(\mathcal{E}_m) = 1.$$

In particular, for $m \gg 0$, $h^0(\mathcal{E}_m) > 0$.

Proof. We will construct the sequence $\mathcal{E}_m$ inductively as follows. Put $\mathcal{E}_0 = \mathcal{E}$ and fix an $\mathcal{O}(2)$-Ulrich sheaf $\mathcal{F}$ on $\mathbb{P}^2$. Now, assume we have constructed $\mathcal{E}_0, \cdots, \mathcal{E}_m$ such that $\mathcal{E}_i$ is δ-Ulrich with respect to $\mathcal{O}_X(2^i)$. To construct $\mathcal{E}_{m+1}$ we consider the embedding $X \to \mathbb{P}^N$ determined by $\mathcal{O}_X(2^m)$. Let $\pi : X \to \mathbb{P}^2$ be a finite map obtained as the compostion of i with a general linear projection $\mathbb{P}^N \dashrightarrow \mathbb{P}^2$. Define $\mathcal{E}_{m+1} = \mathcal{E}_m \otimes \pi^*\mathcal{F}$. By Lemma 4.6, $\pi_*(\mathcal{E}_m \otimes \pi^*\mathcal{F}) = \pi_*(\mathcal{E}_m) \otimes \mathcal{F}$ is δ-Ulrich for $\mathcal{O}(2)$ since $\pi_*\mathcal{E}_m$ is δ-Ulrich for $\mathcal{O}(1)$ and $\mathcal{F}$ is Ulrich for $\mathcal{O}(2)$. Thus $\mathcal{E}_{m+1}$ is δ-Ulrich for $\pi^*\mathcal{O}(2) = \mathcal{O}_X(2^{m+1})$. Moreover,

$$\chi(\mathcal{E}_{m+1}) = \mathrm{rk}(\mathcal{F})(\chi(\mathcal{E}_m) + 3\mathrm{rk}(\pi_*(\mathcal{E}_m))) = \mathrm{rk}(\mathcal{F})(\chi(\mathcal{E}_m) + 3\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))).$$

Since $\deg(\mathcal{O}_X(2^{m+1})) = 4\deg(\mathcal{O}_X(2^m))$, we can write

$$\frac{\chi(\mathcal{E}_{m+1})}{\mathrm{rk}(\mathcal{E}_{m+1})\deg(\mathcal{O}_X(2^{m+1}))} = \frac{1}{4} \cdot \frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} + \frac{3}{4}.$$

Now it is clear that

$$\lim_{m\to\infty} \frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} = 1.$$

On the other hand we have

$$\frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} \leq \alpha(\mathcal{E}_m, \mathcal{O}_X(2^m)) \leq 1$$

and the Theorem follows immediately. □

Remark 4.8. Suppose that $\mathcal{E}$ is a δ-Ulrich sheaf for $\mathcal{O}(1)$ and $\mathcal{F}$ is an $\mathcal{O}(2)$-Ulrich sheaf on $\mathbb{P}^2$. A calculation similar to those in the proof of Lemma 4.6 shows that

$$h^1(\mathcal{E} \otimes \mathcal{F}(-2)) = \mathrm{rk}(\mathcal{F})h^1(\mathcal{E}(-1))$$

Hence

$$\frac{h^1(\mathcal{E} \otimes \mathcal{F}(-2)}{\mathrm{rk}(\mathcal{E} \otimes \mathcal{F})} = \frac{h^1(\mathcal{E}(-1))}{\mathrm{rk}(\mathcal{E})}.$$

So while $\mathcal{E} \otimes \mathcal{F}$ is closer to being $\mathcal{O}(2)$-Ulrich than $\mathcal{E}$ is to being $\mathcal{O}(1)$-Ulrich as measured by $\alpha(-)$, it is no closer at all by this other measure.

Remark 4.9. The minimum rank of an $\mathcal{O}(2)$-Ulrich bundle on $\mathbb{P}^2$ is two. So the ranks of the sheaves $\mathcal{E}_m$ in Theorem 4.7 are growing exponentially.

4.3 Intermediate Cohomology Modules

Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Our last result describes the structure of the graded module $H^1_*(\mathcal{E})$ in a way that refines Corollary 4.2. First we need a definition.

Definition 4.10. Let S be a standard graded ring and M a finitely generated S module. We say that M has the *Weak Lefschetz Property* [MN13] if there is a linear element $z \in S_1$ such that each multiplication map $\mu_z : M_i \to M_{i+1}$ has maximum rank.

Proposition 4.11. *The graded module* $\mathrm{H}^1_*(\mathcal{E})$ *over the graded ring* $S_X = \mathrm{H}^0_*(\mathcal{O}_X)$ *has the Weak Lefschetz property. Moreover, the following inequalities hold:*

$$\begin{aligned} h^1(\mathcal{E}(i)) &\leq h^1(\mathcal{E}(i+1)) \quad (i \leq -2) \\ h^1(\mathcal{E}(i)) &\geq h^1(\mathcal{E}(i+1)) \quad (i \geq -2) \end{aligned}$$

Proof. Let $H \subset X$ be a hyperplane section (with respect to $\mathcal{O}_X(1)$) such that $\mathcal{E}|_H$ is Ulrich and $z \in \mathrm{H}^0(\mathcal{O}_X(1))$ a defining section. Then consider the long exact sequence

$$\mathrm{H}^0(\mathcal{E}|_H(i+1)) \longrightarrow \mathrm{H}^1(\mathcal{E}(i)) \xrightarrow[\mu_z]{} \mathrm{H}^1(\mathcal{E}(i+1)) \longrightarrow \mathrm{H}^1(\mathcal{E}|_H(i+1)) \qquad (*)$$

on cohomology induced by

$$0 \to \mathcal{E}(i) \to \mathcal{E}(i+1) \to \mathcal{E}|_H(i+1) \to 0.$$

Recall that since $\mathcal{E}|_H$ is Ulrich, we have

$$\mathrm{H}^0(\mathcal{E}|_H(i+1)) = 0,\ (i \leq -2), \quad \text{and} \quad \mathrm{H}^1(\mathcal{E}|_H(i+1)) = 0,\ (i \geq -2).$$

So if $i < -1$, the map μ_z in $(*)$ is injective, and if $i > -3$ it is surjective. $\square$

Remark 4.12. An immediate consequence of Proposition 4.11 is that $\mathrm{H}^1_*(\mathcal{E})$ is generated in degree at most -2. We show this is the best possible statement by exhibiting for each $s \geq 2$ a $\delta-$Ulrich sheaf $\mathcal{E}_s$ such that $\mathrm{H}^1_*(\mathcal{E}_s)$ has a generator in degree $-s$. Consider the simplest of the varieties discussed in Remark 3.6, i.e. a smooth quadric surface $X \subseteq \mathbb{P}^3$. Let L_1, L_2 be the line classes which generate $\mathrm{Pic}(X)$. Then $H = L_1 + L_2$ and $H^\perp$ is generated by $L_1 - L_2$. For each $s \in \mathbb{Z}$, the line bundle $\mathcal{E}_s := \mathcal{O}_X(sL_1 + (1-s)L_2)$ is δ-Ulrich, and fails to be Ulrich precisely when $s \neq 0, 1$. For $s \geq 2$ and $k \in \mathbb{Z}$ we have

$$h^1(\mathcal{E}_s(-s+k)) = h^1(\mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1-2s+k)) = \begin{cases} (k+1)(2s-k-2), & 0 \leq k \leq 2s-3 \\ 0 & \text{otherwise} \end{cases}$$

References

[AFO] Marian Aprodu, Gavril Farkas, and Angela Ortega. Minimal resolutions, Chow forms and Ulrich bundles on K3 surfaces. *J. Reine. Angew. Math.*, to appear.

[BHS88] Jörgen Backelin, Jürgen Herzog, and Herbert Sanders. Matrix factorizations of homogeneous polynomials. In *Algebra—some current trends (Varna, 1986)*, volume 1352 of *Lecture Notes in Math.*, pages 1–33. Springer, Berlin, 1988.

[BHU87] Joseph P. Brennan, Jürgen Herzog, and Bernd Ulrich. Maximally generated Cohen-Macaulay modules. *Math. Scand.*, 61(2):181–203, 1987.

[BHU91] J. Backelin, J. Herzog, and B. Ulrich. Linear maximal Cohen-Macaulay modules over strict complete intersections. *J. Pure Appl. Algebra*, 71(2-3):187–202, 1991.

[Chi78] Lindsay N. Childs. Linearizing of n-ic forms and generalized Clifford algebras. *Linear and Multilinear Algebra*, 5(4):267–278, 1977/78.

[CHW] Izzet Coskun, Jack Huizenga, and Matthew Woolf. Equivariant Ulrich Bundles on Flag Varieties. *preprint*, `http://arxiv.org/abs/1507.00102`.

[CMR15] L. Costa and R. M. Miró-Roig. $GL(V)$-invariant Ulrich bundles on Grassmannians. *Math. Ann.*, 361(1-2):443–457, 2015.

[CMRPL12] Laura Costa, Rosa M. Miró-Roig, and Joan Pons-Llopis. The representation type of Segre varieties. *Adv. Math.*, 230(4-6):1995–2013, 2012.

[ESW03] David Eisenbud, Frank-Olaf Schreyer, and Jerzy Weyman. Resultants and Chow forms via exterior syzygies. *J. Amer. Math. Soc.*, 16(3):537–579, 2003.

[FPL] Daniele Faenzi and Joan Pons-Llopis. The CM representation type of projective varieties. *preprint*, `http://arxiv.org/abs/1504.03819`.

[Han99] D. Hanes. Special Conditions on Maximal Cohen-Macaulay Modules, and Applications to the Theory of Multiplicities. University of Michigan Ph. D. thesis, 1999.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[Jar06] Marcos Jardim. Instanton sheaves on complex projective spaces. *Collect. Math.*, 57(1):69–91, 2006.

[KMSa] Rajesh S. Kulkarni, Yusuf Mustopa, and Ian Shipman. The characteristic polynomial of an algebra and representations. *to appear.*

[KMSb] Rajesh S. Kulkarni, Yusuf Mustopa, and Ian Shipman. Ulrich Sheaves and Higher-Rank Brill-Noether Theory. *preprint*, `http://arxiv.org/abs/1312.2514`.

[MN13] Juan Migliore and Uwe Nagel. Survey article: a tour of the weak and strong Lefschetz properties. *J. Commut. Algebra*, 5(3):329–358, 2013.

[MR13] Rosa M. Miró-Roig. The representation type of rational normal scrolls. *Rend. Circ. Mat. Palermo (2)*, 62(1):153–164, 2013.

[Pap00] Christopher J. Pappacena. Matrix pencils and a generalized Clifford algebra. *Linear Algebra Appl.*, 313(1-3):1–20, 2000.

[Pro87] Claudio Procesi. A formal inverse to the Cayley-Hamilton theorem. *J. Algebra*, 107(1):63–74, 1987.

[Rob69] Norbert Roby. Algèbres de Clifford des formes polynomes. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A484–A486, 1969.

[VdB87] M. Van den Bergh. Linearisations of binary and ternary forms. *J. Algebra*, 109(1):172–183, 1987.